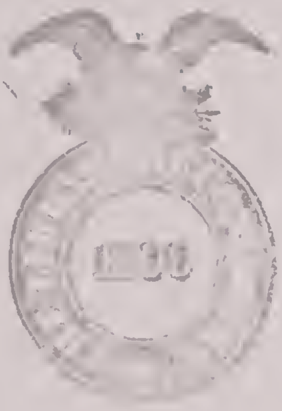

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MODERN
HIGH SCHOOL ALGEBRA

BY

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PREFACE

The most notable undertaking in the field of instruction in mathematics in the past decade has been the work of the National Committee on Mathematical Requirements, culminating in its summary report, recently issued, and in its final report which will appear in a short time. One immediate result of this undertaking has been a revision of the definitions of the units in high school mathematics by the College Entrance Requirements Board. The extent to which the Wells and Hart *High School Algebra*, written eleven years ago, harmonizes with these recent reports is most gratifying.

This *Modern High School Algebra* incorporates, first of all, the changes suggested by the eleven added years of experience of its author in teaching the subject to boys and girls of high school grade; and secondly, other changes which are suggested by these recent reports.

It is a new book. While, naturally, it is based upon the Wells and Hart *High School Algebra*, the text has been rewritten wherever any improvement could be made in the instruction or expression. The explanations in the text are often more complete than those of the *High School Algebra,* — and those represented a high-water mark eleven years ago. This is done to help teachers — not to endeavor to supersede them. The examples are all new, and so, too, are almost all the problems. Enough examples and problems are given in the text so that teachers will not need to seek drill material from other sources.

The new features of this text are:

1. **Increased emphasis on the formula.**

   In Chapter I, the formula is made the means of transition from arithmetic to algebra. It is emphasized throughout the text as its importance demands. (See pages 28, 61, 115, 242, etc.)

2. **Abundant use of geometrical material** both as a means of instruction and as problem material.

   The geometrical concepts and facts are taught inductively. **This is intuitive geometry.** Only those geometrical facts have been used which can be unified with the algebra. **This is unified general mathematics** — not mere mixed mathematics. (See pages 4–8, 22–25, 61–62, 76–78, 122, 442, etc.)

3. **The function concept** has been foreshadowed in various forms in the text, — and that seems about all that experience with beginning pupils warrants.

> Besides the obvious contribution to this concept of the work with the formula and the graph, see the development of the idea of *dependence* or *relationship* as in Examples 4–7, pages 2–3; § 23; page 241; page 282; page 406.

4. **The elimination from the suggested minimum course** for the first year of H. C. F. (page 153); "signs in fractions" (page 181); the theory of proportion, which belongs more properly in a course in geometry (pages 203–207); literal equations (pages 198, 241, 278); linear equations in $1/x$ and $1/y$ (page 243); sets of equations having three unknowns (page 245); operations with surd expressions (page 257); simultaneous quadratics (page 279); imaginaries (page 285); generalized forms of special products, and factoring (page 288).

These topics appear in sections marked **supplementary topics.** Omission of them leaves **a minimum course which every school can master in a single year.** This minimum course contains all the topics *in algebra* recommended by the National Committee and by the College Entrance Requirements Board. It is **the briefest minimum course in algebra** that has been proposed for the first course. The supplementary sections provide material needed for certain local courses of study and for teachers who need extra material for especially bright pupils.

> The author cannot agree with the National Committee on the desirability or necessity of omitting long division of polynomials and square root of polynomials. These do not present any difficulties for pupils, and they are good teaching material.

Similarly, in the part of the text designed for the third semester of algebra, certain sections are designated as supplementary.

5. A brief chapter on **the trigonometry of the right triangle** is included. (See page 416.)

6. The **consistent use of "charts"** in the solution of problems is suggested.

> These charts have been found a great help to pupils who are learning to solve problems (pages 85, 87, 88, etc.).

7. Lists of classified examples and problems at the back of the text for special drill purposes, beginning on page 426.

8. **Timed tests** on fundamentals on page 451. These tests will enable those who are using this text to compare results in an interesting way with others who are using the same text.

9. Summary of words and processes which the pupils should know at the end of the course. (See page 460.)

The following features of this text **have met with distinct approval for a decade** by users of Wells and Hart texts, and are sanctioned either explicitly or implicitly by the report of the National Committee.

1. **Each topic taught is used in the solution of equations.**

   This gives a motive for the study of each topic, permits gradation of material, and emphasizes the equation. (See §§ 14, 43, 47, 55, etc.)

2. **Problems receive special attention.**

   The different kinds are introduced gradually, and are taught carefully when they first appear. (See pages 85, 87, 101, 145, etc.) They appear repeatedly thereafter in miscellaneous lists.

3. **Factoring is cut down to its simplest possible form** by omitting from the first treatment all types not used in the first course in algebra.

   The omitted types are taught in Chapter XV. This chapter provides a desirable review of factoring from a higher point of view, early in the third semester of algebra. Teachers will find that study of this chapter late in the course will prove effective preparation for any required examination.

4. **Thoughtful solutions of equations and examples are encouraged** by subordinating "transposition," "clearing of fractions," etc.

   Observe the omission of these processes until page 104 and the suggestion in § 79 that they can be omitted altogether. Notice the convenient symbols A, S, M, and D, used in solving equations. (See pages 48, 59, etc.)

5. **Attempt is made to maintain and increase skill in arithmetic** by emphasizing evaluation (pages 2, 26); by checking by substitution; by the use of fractional and decimal coefficients (pages 8, 9, etc.); by expressing roots decimally (page 261).

6. **Graphical work is taught and used where it is most helpful**, just before simultaneous equations. Thereafter it is used whenever it aids in giving instruction in some algebraic process. (See pages 200–224; 282–287; 307; 309–315.)

MARCH, 1923.

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To the Teacher:

This text is designed to aid you in promoting mastery of the beginnings of algebra by your pupils. Your attention is directed to the minimum course which remains when the supplementary topics are omitted. This minimum course is supplemented by other lists of drill material at the back of the book, by timed tests, and by a list of words and processes which pupils should know at the end of the year. Use this material as means of mastering the minimum course. **Insist upon mastery of it.** Some pupils can accomplish this by working a few examples; others will need to work many examples. Some can do it quickly; others will require more time. Have your pupils check off in the list at the back of the book each new word and process as you encounter it in your course.

The supplementary material should be considered optional. If necessary, agree that only certain types of problems shall be mastered by those who get the lowest passing mark. Let the higher marks be regarded as rewards for mastering other types of problems and for mastering some of the supplementary material. This point of view of mastery is discussed most fruitfully by Professor Morrison, of the University of Chicago, in the *School Review* of January, February, and March, 1921.

To the Pupil:

Let your motto be: "I can do it."

ALGEBRA

I. LITERAL NUMBERS — FORMULÆ

1. Letters are used to represent numbers, in order to abbreviate rules.

Thus, $P = B \times R$

expresses the rule

the percentage equals the base multiplied by the rate.

B represents the base, R the rate per cent, and P the percentage.

$$\text{If } B = 2500, \text{ and } R = 5\,\%,$$
$$P = 2500 \times .05 = 125.$$

A letter used to represent a number is called a literal number.

The use of literal numbers is one characteristic of algebra.

2. Signs of multiplication. The symbol × is not used much in algebra as the sign of multiplication because it is like the letter x. A dot, ·, placed above the line between two numbers means that the numbers are to be multiplied together.

Thus, $6 \cdot 7$ means 6 times 7, or 42.
$a \cdot b$ means a times b.

Even the dot is omitted when one or both of the numbers are literal numbers.

Thus, $6\,a$ means 6 times a. It is read "six a."
xt means x times t. It is read "ex tee."

Historical Note. — The symbol × was first used by an Englishman, Oughtred, about 1631. The symbol • was introduced by Leibnitz in 1693. Multiplication was indicated in Hindu and Italian books by writing the factors side by side, as early as the thirteenth century.

Look up the biographies of such men when they are mentioned in the historical notes.

3. The **product** of two or more numbers is the result obtained by multiplying together the numbers; the numbers themselves are the **factors** of the product. *Every factor of a product is an exact divisor of the product.*

Thus, 5 and 8 are factors of 40.

EXERCISE 1

1. What does $5m$ mean? $8x$? $7t$?

2. How much is, or what is the value of $10x$ when x is 7? Also when x is 13? When x is 2.5?

3. What is the value of $20y$ when y is 3? When y is .5? When y is $\frac{1}{4}$?

4. Make a table like the one at the right, or complete this table orally. Opposite each value of x given in the first column, write or give the value of $6x$ in the second column.

When x is	Then $6x$ is
3	
8	
11	
$\frac{1}{3}$	2
.5	

Thus, opposite $\frac{1}{3}$ we write 2, because when x is $\frac{1}{3}$, $6x$ is $6 \times \frac{1}{3}$, or 2.

5. See how rapidly you can complete the following tables:

a

When y is	Then $8y$ is
7	
12	
2.5	
$\frac{1}{4}$	
$\frac{5}{8}$	

b

When m is	Then $24m$ is
$\frac{1}{3}$	
$\frac{3}{4}$	
$\frac{5}{6}$	
$\frac{7}{8}$	
$\frac{11}{12}$	

c

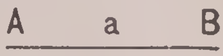
When t is	Then $18t$ is
5	
9	
.6	
$\frac{1}{9}$	
$\frac{2}{3}$	

6. *a.* What are the values of $9y$ when y is 3, 5, 6, 10, 20?

b. As y increases (becomes greater), what happens to $9y$?

7. *a.* What are the values of $10\,A$ when A is 15, 12, 10, 9, 4?
b. As A decreases (becomes smaller), what happens to $10\,A$?

8. Suppose that a represents the length of segment AB. What does $3\,a$ then represent?



9. *a.* How much do 5 doz. eggs cost at 40¢ per dozen? At 50¢ per dozen? At x cents per dozen?
b. How much is $5\,x$ cents when x is 60?

10. A certain number is represented by n. What represents the number 4 times as large? 11 times as large?

11. If b is the number of feet in the base of a rectangle, and 5 is the number in the altitude, what is the area?

12. If h is the number of feet in the altitude of a rectangle, and if the base is 3 times as long, how many feet are there in the base?

13. *a.* What is the interest on $100 for 1 yr. at 5%? for 3 yr.?
b. What is the interest on P dollars for 1 yr. at 5%? for 3 yr.?

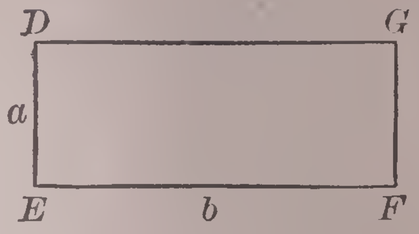
14. *a.* An article cost $5. It was sold so that the profit was 10% of the cost. What was the profit?
b. An article cost C dollars. It also was sold so that the profit was 10% of the cost. What was the profit?
c. In part *b*, how much would this profit be if $C = 20$?

15. *a.* How much do 8 tons of coal cost at $12 per ton?
b. How much do 8 tons of coal cost at p dollars per ton?
c. What is the value of the result in part *b* when $p = 10$?

4. A **formula** is a rule of computation expressed by means of arithmetical and literal numbers, connected by mathematical signs which tell what must be done with the numbers.

Thus, in § 1, $P = BR$ is a formula. It is called the percentage formula.

The use of literal numbers in formulæ is one of their most important uses.



5. **Making and using some important formulæ.**

a. **The formula for the area of a rectangle.**

You know that

> *The area of a rectangle equals the product of its base and altitude.*

Thus, if the base is 8 in., and the altitude is 15 in., the area is 8 × 15 or 120 sq. in.

If a = the number of units of length in the altitude,
b = the number of the same units in the base,
and A = the number of the corresponding surface units in the interior of the rectangle,
then

$$A = ab.$$

This formula for the area of a rectangle is used as follows:

Example. — Find the area of the rectangle whose base is $2\frac{1}{2}$ ft. and whose altitude is 15 in.

Solution. — 1. The formula is $A = ab$.
2. $a = 15$ in.; $b = 2\frac{1}{2}$ ft. $= 30$ in.; $A = ?$
3. $\therefore\ A = 15 \times 30 = 450$ sq. in.

NOTE 1. — In Step 3, the symbol $\therefore$ means "therefore."

NOTE 2. — To "explain" Step 3, say, "I substituted 15 for a and 30 for b in the formula." "Substitute" means "put in place of."

EXERCISE 2

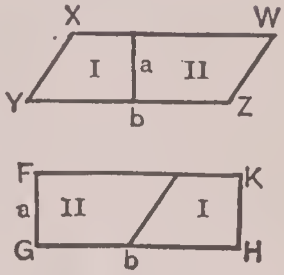
Solve the following problems, using the form illustrated above.

1. What is the area of the rectangle whose base is 13 ft. and whose altitude is 8 ft.?

2. Find the area of the rectangle whose base is 3 yd. and whose altitude is 4 ft.

3. Find the area of the rectangle whose base is 10 ft. 6 in. and whose altitude is 8 ft.

b. **The formula for the area of a parallelogram.**



The figure $XYZW$ is a parallelogram.

a. How does $XYZW$ compare in area with $FGHK$?

b. If the altitude of the parallelogram is 5 in. and the base is 12 in., what is the area of the rectangle $FGHK$? What, then, must be the area of the parallelogram $XYZW$?

c. If the altitude of the parallelogram were 10 in. and the base 16 in., what would be its area?

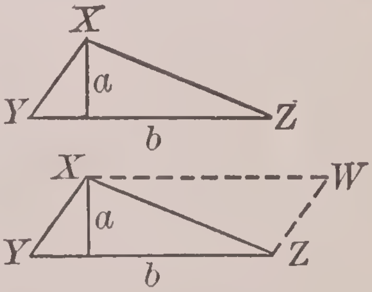
d. Evidently the area of a parallelogram is the product of its base and altitude.

If a = the number of units of length in the altitude,
b = the number of the same units of length in the base,
and A = the number of corresponding surface units inside the parallelogram,

then

$$A = ab.$$

c. **The formula for the area of a triangle.**



The figure XYZ is a triangle.

Notice that triangle XYZ incloses one half as much surface as parallelogram $XYZW$.

Since the area of the parallelogram is the product of its base and altitude, then *the area of the triangle must be one half the product of its base and altitude.*

If the letters a, b, and A have the same meaning as above, then the formula for the area of a triangle is

$$A = \frac{1}{2} ab.$$

Example. — What is the area of a triangle whose base is $2\frac{1}{3}$ yd. and whose altitude is 2 ft.?

Solution. — 1. The formula is $A = \frac{1}{2}ab$.

2. $b = 2\frac{1}{3}$ yd. $= 7$ ft.; $a = 2$ ft.; $A = ?$

3. Substituting in the formula,

$$A = \frac{1}{2} \cdot 7 \cdot 2 = 7 \text{ sq. ft.}$$

EXERCISE 3

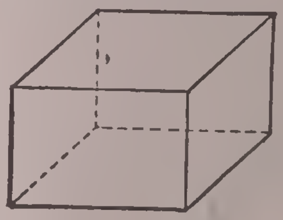
1. Find the area of a parallelogram whose base is 18 ft. and whose altitude is 5 ft. 4 in.

2. Find the area of a triangle whose base is $12\frac{1}{2}$ ft. and whose altitude is 9 ft.

3. Find the area of a triangle whose base is 3 ft. 4 in. and whose altitude is 15 in. (Express each in terms of inches or feet.)

4. What is the area of the parallelogram whose base is $2\frac{1}{3}$ ft. and whose altitude is 20 in.?

5. The figure at the right is a **rectangular parallelopiped.** You know that its volume equals its length multiplied by its width, multiplied by its height. If the length, width, and height are represented by l, w, and h, respectively, and the volume by V, then



$$V = lwh$$

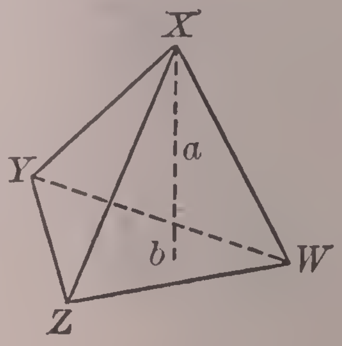
is the **formula for the volume of a rectangular parallelopiped.**

By this formula, find the volume of the rectangular parallelopiped whose length is 9 in., width 12 in., and height 7 in.

6. Find the volume of the rectangular parallelopiped whose length is 30 ft., width 6 yd., and height 15 ft.

7. The figure at the right is a **pyramid.**

The volume of a pyramid is one third of the product of its base and altitude. If a = the number of units of length in the



altitude, b = the number of the corresponding surface units in the base, and V = the number of corresponding cubic units in the pyramid, then

$$V = \tfrac{1}{3}ab$$

is the formula for the volume of a pyramid.

Find the volume when the altitude is 12 in. and the base contains 10 sq. in.

8. Find the volume of a pyramid whose altitude is 5 yd. and whose base contains 35 sq. ft.

9. *a.* **The formula for finding the simple interest on a sum of money is $I = PRT$.** Express this formula in words.

b. Find I when P = \$3600, R = 5%, and T = 4 yr.

10. Find the interest on \$2500 invested at 6% for 3 yr.

11. Find the simple interest on \$4600 invested at 8% for 2 yr. 6 mo.

12. *a.* The rule for finding the circumference of a circle is to multiply the radius by twice the number 3.1416, which is called pi. If r = the length of the radius, π = the number 3.1416, and C = the circumference, this rule is expressed by the formula

$$C = 2\pi r.$$

b. Find the circumference of the circle whose radius is 5 in.

13. $3\frac{1}{7}$ instead of 3.1416 may be used as the value of π, wherever the number π appears. Using this value of π, find the circumference of the circle whose radius is 21 in.

14. What is the length (circumference) of the circle whose diameter is 14 ft.?

15. What is the simple interest on \$650 at 7% for 9 mo.? (Remember to express 9 mo. as a part of a year.)

16. By the formula $Z = 2\pi rh$, find Z when $r = 14$, $h = 16$, and $\pi = 3\frac{1}{7}$.

17. By the formula $S = \frac{1}{2}CL$, find S when $C = 32.4$ and $L = 15$.

18. By the formula $S = \pi r L$, find S when $r = 28$, $L = 40$, and $\pi = 3\frac{1}{7}$.

19. By the formula $L = 2\pi r h$, find L when $\pi = 3.1416$, $r = 8$, and $h = 10$.

20. By the formula $F = \frac{9}{5}C$, find F when $C = 45$.

6. Memorize the following formulæ which you used in the preceding exercise.

The formula for:

a. The area of a rectangle is $A = ab$.
b. The area of a parallelogram is $A = ab$.
c. The area of a triangle is $A = \frac{1}{2}ab$.
d. The volume of a rectangular parallelopiped is $V = lwh$.
e. The volume of a pyramid is $V = \frac{1}{3}ab$,
f. The circumference of a circle is $C = 2\pi r$.
g. Simple interest is $I = PRT$.

7. An important law of multiplication.

In arithmetic, $2 \times 3 \times 4 = (2 \times 3) \times 4 = 6 \times 4 = 24$.
Similarly, in algebra, $5 \times 4m = (5 \times 4) \times m = 20m$.

NOTE. — The symbol ( ) is called *parentheses*. The numbers inside are to be combined before using them in any other addition, subtraction, multiplication, etc.

EXERCISE 4

Give the following products orally, or write only the results.

1. $7 \cdot 8x$
2. $3 \cdot 11b$
3. $9 \cdot 12m$
4. $8 \cdot 15t$
5. $12 \cdot 8w$
6. $11 \cdot 7t$
7. $6 \cdot 13v$
8. $9 \cdot 15z$
9. $4 \cdot \frac{3}{4}x$
10. $5 \cdot \frac{2}{5}y$
11. $6 \cdot \frac{5}{6}z$
12. $12 \cdot \frac{1}{2}w$
13. $20 \cdot \frac{3}{5}m$
14. $24 \cdot \frac{5}{6}p$
15. $36 \cdot \frac{2}{9}r$
16. $40 \cdot \frac{3}{8}s$

17. *a.* One number is represented by $3x$. What represents the number twice as large? Five times as large?

b. How much is each of these numbers when x is 2?

18. *a.* Let f represent the first of certain three numbers. If the second is 5 times the first, and the third is 3 times the second, represent each of them.

b. Find the value of each of these numbers when f is 4.

19. The area of a certain rectangle is $17x$ square inches. What is the area of a parallelogram twice as large?

20. *a.* Edward is $3y$ years of age. Will is 5 times as old. How old is Will?

b. What is the age of each, if y is 2?

8. An important law of division.

Since $2 \cdot 3a = 6a$, then $6a \div 2 = 3a$.

Similarly, since $5 \cdot 8x = 40x$, then $40x \div 5 = 8x$

Also, $20x \div 5 = 4x$; $18y \div 9 = 2y$.

EXERCISE 5

Give orally or write only the quotients:

1. $25t \div 5$

2. $32r \div 8$

3. $60w \div 12$

4. $24x \div 4$

5. $36x \div 36$

6. $29y \div 29$

7. $39t \div 39$

8. $2.5z \div 2.5$

9. $84p \div 12$

10. $16m \div 16$

11. $.5w \div .5$

12. $\frac{1}{2}z \div \frac{1}{2}$

13. What part of $32x$ is:

a. $8x$? *b.* $16x$? *c.* $4x$? *d.* x?

14. What part of $28t$ is:

a. $2t$? *b.* $7t$? *c.* $4t$? *d.* t?

15. *a.* By what must you divide $6x$ to get x?

b. By what must you divide $18r$ to get $2r$?

c. By what must you divide $25z$ to get z?

d. By what must you divide $2.5x$ to get x?

e. By what must you divide $.56y$ to get y?

HISTORICAL NOTE. — The symbol ÷ was introduced by John Pell, who lived during the seventeenth century.

9. An equation expresses an equality of numbers. A formula is one kind of equation.

Thus, $A = 3b$ is an equation.

It states that the number A is 3 times the number b.

Again, $3x + 11 = 26$ is an equation.

It states that the result of adding 11 to 3 times a certain number, x, is 26.

The numbers on the right side of the equality sign form the **right member** or the **right side of the equation**; the ones on the left side of the equality sign form the **left member** or **left side of the equation.**

10. Solving an equation.

Consider the equation $3x + 11 = 26$.

When $x = 5$, $3x + 11$ is $3 \cdot 5 + 11$ or $15 + 11$.

This does indeed equal 26. **5 is said to satisfy the equation.**

5 is called a **root** of the equation.

But when x is any other number, $3x + 11$ does not equal 26.

Thus, if $x = 8$, $3 \cdot 8 + 11 = 24 + 11 = 35$.

In such an equation, *the letter represents an unknown number;* often for this reason, the letter itself is called the unknown number.

EXERCISE 6

1. Does 6 satisfy the equation $3x + 8 = 26$?
2. Does 10 satisfy the equation $4y + 7 = 47$?
3. Does 3 satisfy the equation $5z + 11 = 31$?
4. Does 6 satisfy the equation $5z + 11 = 31$?
5. What number does satisfy the equation $5z + 11 = 31$?

11. Rule I used in solving equations.

Suppose the weight and the sugar just balance on the scales pictured at the right.

Then one half the sugar will also balance one half the weight. It is equally clear that one half the sugar will not balance one third of the weight.

Similarly, in an equation like $10\,x = 80$,
if both sides are divided by 2, the results must be equal; that is,

$$5\,x = 40.$$

But if one side is divided by 2, and the other side is divided by some other number, the results will not be equal.

Rule. — Both members of an equation may be divided by the same number without destroying the equality.

Example 1. — Solve the equation $17\,y = 391$.

Solution. — 1. y can be got from $17\,y$ by dividing $17\,y$ by 17. Then, in order to keep the two sides equal, both sides must be divided by 17.

2. Then, $y = 23$.

$$\begin{array}{r} 23 \\ 17\overline{)391} \\ \underline{34} \\ 51 \\ \underline{51} \end{array}$$

In order to abbreviate the written explanations of solutions, the symbol D will be used as follows:

D_{17} will mean "divide both sides of the preceding equation by 17."

Example 2. — Solve the equation $6.5\,w = 11.7$.

Solution. — 1. $6.5\,w = 11.7.$

2. $D_{6.5}$ $w = 1.8.$

$$\begin{array}{r} 1.8 \\ 6\overset{\times}{.}5.\overline{)11\overset{\times}{.}7.0} \\ \underline{65} \\ 520 \\ \underline{520} \end{array}$$

Note. — In Step 2, what does $D_{6.5}$ mean? Notice that $6.5\,w \div 6.5 = w$; and that $11.7 \div 6.5 = 1.8$. Also, that $w = 1.8$ by Rule I.

EXERCISE 7

Solve the following equations, using the form of solution illustrated in Example 2.

1. $11\,x = 286$
2. $15\,y = 285$
3. $17\,z = 476$
4. $22\,W = 814$
5. $9\,A = 2583$
6. $14\,m = 133$
7. $27\,p = 221.4$
8. $35\,r = 539$
9. $44\,s = 158.4$
10. $56\,c = 420$
11. $32\,w = 528$
12. $8.4\,y = 168$
13. $.12\,m = 24$
14. $1.35\,x = 911.25$
15. $.75\,p = 1800$

NOTE. — Some of the problems which follow can be done by arithmetic alone. Do them algebraically even if the arithmetical solution appears easier to you at the present time.

16. If a certain number be multiplied by 17, the product is 144.5. What is the number?

Solution. — 1. Let $n =$ the number.
2. Then $17n =$ 17 times the number.
3. Then $17n = 144.5$, according to the statement of the problem.

Finish this problem by dividing both sides by 17.

17. If a certain number be multiplied by 13, the product is 247. What is the number?

18. If a certain number be multiplied by 8.5, the result is 221. What is the number?

19. How many articles at $1.75 each can be bought for $40.25?

Solution. — 1. Let $n =$ the number of articles.
2. $\therefore n \cdot \$1.75$ or $1.75n =$ the cost of the articles.
3. $\therefore 1.75n = 40.25$.
(Complete this problem.)

20. How many articles at $2.15 each can be bought for $38.70?

21. What number, multiplied by 2.7, gives 9.45 as product?

22. How many miles per hour did an auto travel which went 172 miles in 8 hours?

23. At what price must an article be purchased if 16 of them cost $13.60?

24. If a certain number be multiplied by 12.5, the product is 217.25. What is the number?

25. How much must each of 25 persons contribute to make up a fund of $156.25, if all contribute equally?

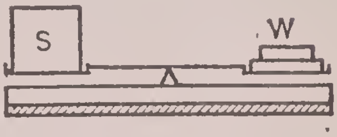
26. The product of two numbers is 499.5. One of them is 18.5. What is the other?

27. How much must a boy save each month to have $6.75 at the end of 15 months?

12. Rule II used in solving equations.

Suppose the sugar S balances the weight W on the scales.

Then double the amount of sugar will also balance double the amount of weight.



It is equally clear that double the amount of sugar will not balance three times the amount of weight.

These facts help you understand the

Rule. — Both members of an equation may be multiplied by the same number without destroying the equality.

Example 1. — Solve the equation $\frac{3}{8}x = 36$.

Solution. — 1. $\frac{3}{8}x = 36$.

2. We decide to get rid of the fraction $\frac{3}{8}$. If we multiply it by 8, then $8 \times \frac{3}{8}$ is 3.

Then, in order to keep the two sides equal, we multiply both sides of the equation by 8.

3. Then, $3x = 288$.

4. D_3 $x = 96$.

Check. — Does $\frac{3}{\cancel{8}} \times \overset{12}{\cancel{96}} = 36$? Yes.

Note. — Observe that $8 \cdot \frac{3}{8}x = 3x$; that $8 \cdot 36 = 288$; that $3x = 288$ by Rule II. All this is done mentally when passing from Step 2 to Step 3.

Instead of writing in full

"multiply both sides of the previous equation by 8,"

we shall write M_8.

Example 2. — Solve and check the equation $92.4 = \frac{7}{3}w$.

Solution. — 1. $92.4 = \frac{7}{3}w$.

2. M_3 $277.2 = 7w$,

or $7w = 277.2$.

3. D_7 $w = 39.6$.

Check. — Substituting 39.6 for w in the original equation,

does $92.4 = \frac{7}{\cancel{3}} \times \overset{13.2}{\cancel{39.6}}$? Yes.

$$\begin{array}{r} 13.2 \\ 7 \\ \hline 92.4 \end{array}$$

Note. — What does M_3 mean in Step 2 above? D_7 in Step 3?

EXERCISE 8

Solve the following equations, using the form of solution illustrated in Example 2.

1. $\frac{5}{8}x = 40$
2. $\frac{7}{9}y = 84$
3. $\frac{3}{11}z = 54$
4. $\frac{5}{12}t = 30$
5. $\frac{3}{14}w = 9$
6. $\frac{5}{3}v = 80$
7. $\frac{7}{4}r = 77$
8. $\frac{8}{5}s = 120$
9. $\frac{9}{8}m = 189$
10. $\frac{11}{6}c = 99$
11 $12 = \frac{4}{15}A$
12. $36 = \frac{9}{5}B$
13. $\frac{5}{9}y = 16$
14. $\frac{7}{8}x = 28.7$
15. $\frac{7}{6}m = 50.75$

16. Fifteen ninths of a certain number is 75. What is the number?

Solution. — 1. Let $n =$ the number.
2. $\therefore \frac{15}{9}n = 75.$
Complete the solution.

17. Five ninths of a certain number is 55. What is the number?

18. Seven thirds of a certain number is 112. What is the number?

19. Eight fifths of the cost of a certain lot was $3000. What was the cost of the lot?

20. Three eighths of the area of Lake Michigan is 8625 sq. mi. What is the area of Lake Michigan?

21. When asked her age, a girl replied, seven fifths of it is 21. What was her age?

22. Four sevenths of the members of a certain class were boys. If there were 28 boys, what was the total membership of the class?

23. The selling price of a certain article was five thirds of its cost. What must have been its cost if it sold for $3.50?

24. Five eighths of the cost of the Suez Canal was $62,500,000. What was the cost of the canal?

25. Nineteen twentieths of the population of the United States in 1920 was 100,425,089. What was the population in that year?

13. A second important law of multiplication.

$$\begin{aligned} \text{You know that } 2 \times 3 \times 4 &= 6 \times 4 = 24; \\ \text{that } 3 \times 2 \times 4 &= 6 \times 4 = 24; \\ \text{that } 2 \times 4 \times 3 &= 8 \times 3 = 24. \end{aligned}$$

That is, $2 \times 3 \times 4 = 3 \times 2 \times 4 = 2 \times 4 \times 3$.

Rule. — The factors of a product may be rearranged in any order before finding the product. ✓

This rule is used in algebra as follows:

$$6 \cdot a \cdot 5 = 6 \cdot 5 \cdot a = 30\,a.$$

$$8 \cdot b \cdot \frac{3}{4} = \overset{2}{\cancel{8}} \cdot \frac{3}{\cancel{4}} \cdot b = 6\,b.$$

The factors should be rearranged and multiplied mentally, however.

EXERCISE 9

Find mentally:

1. $7 \cdot m \cdot 8$

2. $6 \cdot y \cdot 9$

3. $7 \cdot z \cdot 3$

4. $12 \cdot y \cdot \frac{2}{3}$

5. $16 \cdot x \cdot \frac{3}{4}$

6. $18 \cdot z \cdot \frac{5}{6}$

7. $2 \cdot r \cdot 5 \cdot 3$

8. $6 \cdot s \cdot 4 \cdot 5$

9. $15 \cdot t \cdot \frac{1}{3} \cdot 8$

14. The second step in using formulæ.

You are now prepared to find any number in a simple formula like those given in § 6. This is a most valuable mathematical accomplishment.

Example 1. — From the formula $V = lwh$, find w if

$$V = 1456,\ l = 7, \text{ and } h = 16.$$

Solution. — 1. The formula is $V = lwh$.

$$V = 1456;\ l = 7;\ h = 16;\ w = ?$$

2. Substituting in the formula

$$1456 = 7 \cdot w \cdot 16.$$

3. $\therefore$ $$1456 = 112\,w,$$

or $$112\,w = 1456.$$

4. D_{112} $$w = 13.$$

$$\begin{array}{r} 13 \\ 112\overline{)1456} \\ \underline{112} \\ 336 \\ \underline{336} \end{array}$$

Check. — Does $7 \times 13 \times 16 = 1456$? Yes.

$$\begin{array}{r} 13 \\ \underline{7} \\ 91 \end{array} \qquad \begin{array}{r} 16 \\ \underline{91} \\ 16 \\ \underline{144} \\ 1456 \end{array}$$

Example 2. — From the formula $V = \frac{1}{3}\,hb$ find b when $V = 117$ and $h = 13$.

Solution. — 1. Substituting in the formula

$$117 = \tfrac{1}{3} \cdot 13 \cdot b,$$

or $$\tfrac{13}{3}\,b = 117.$$

2. M_3 $$13\,b = 351.$$

3. D_{13} $$b = 27.$$

Check. — Does $\frac{1}{\cancel{3}} \times 13 \times \overset{9}{\cancel{27}} = 117$? Yes.

EXERCISE 10

1. By the formula $V = lwh$, find h when $V = 672$, $l = 12$, and $w = 8$.

2. By the formula $V = lwh$, find w when $V = 2340$, $l = 20$, and $h = 13$.

3. By the formula $V = lwh$, find l when $V = 1360$, $w = 8.5$, and $h = 10$.

4. By the formula $A = lc$, find c when $A = 3309.6$ and $l = 12$.

5. By the formula $S = 2\,\pi rh$, find h when $S = 528$, $\pi = \frac{22}{7}$, and $r = 14$.

6. By the formula $A = \frac{1}{2}\,hb$, find b when $A = 189$ and $h = 14$.

7. By the formula $V = \frac{1}{3}\,hb$, find b when $V = 247$ and $h = 19$.

8. By the formula $V = \frac{1}{3}\,hb$, find h when $V = 232$ and $b = 29$.

9. By the formula $A = \frac{1}{2}ab$, find b when $A = 47.5$ and $a = \frac{19}{3}$.

10. By the formula $C = 2\pi r$, find r when $C = 66$ if $\pi = \frac{22}{7}$.

11. What is the altitude of the rectangle whose area is 594 sq. ft. and whose base is 33 ft.?

Solution. — 1. The formula for the area of a rectangle is

$$A = ab.$$

2. In this problem, $A = 594$; $b = 33$; $a = ?$

Complete the solution as in Examples 1–10.

12. What is the base of the triangle whose area is 651 sq. ft. and whose altitude is 21 ft.?

13. What is the altitude of the parallelogram whose area is 153 sq. yd. and whose base is 8.5 yd.?

14. What is the altitude of the rectangular parallelopiped whose volume is 3240 cu. in., whose width is 15 in., and whose length is 24 in.?

15. What is the altitude of the pyramid whose volume is 396 cu. in. and whose base is 27 sq. in.?

16. What is the base of the triangle whose area is 300 sq. ft., and whose altitude is 15 ft.?

17. What is the base of the parallelogram whose area is 372 sq. ft., and whose altitude is 15.5 ft.?

18. What is the width of the rectangular parallelopiped whose volume is 5400 cu. in., whose altitude is 18 in., and whose length is 25 in.?

19. What is the length of the rectangular parallelopiped whose volume is 2550 cu. ft., whose width is $12\frac{1}{2}$ ft., and whose height is $8\frac{1}{2}$ ft.?

20. What is the base of a pyramid whose altitude is 16 in. and whose volume is 256 cu. in.?

NOTE. — Supplementary examples of the same kind are on page 426.

15. Adding and subtracting numbers having a common factor. Any exact divisor of a number is a **factor** of it.

Thus, 3 is a factor of 21.

A **common factor** of two or more numbers is a factor of each of them.

Thus, 3 is a common factor of 6 and 9.
x is a common factor of $4x$ and $11x$.

The result obtained by adding two or more numbers is called their **sum**. The numbers themselves are called **addends**.

The method of adding numbers which have a common factor is taught in the following exercise.

EXERCISE 11

1. $3 \textit{ times } 7 + 2 \textit{ times } 7 = 5 \textit{ times } 7$
Because $(7 + 7 + 7) + (7 + 7) = (7 + 7 + 7 + 7 + 7)$.

2. $5 \textit{ times } 9 + 3 \textit{ times } 9 = 8 \textit{ times } 9 = ?$

3. $(7 \times 4) + (3 \times 4) = ? \times 4 = ?$

4. $(2 \times 6) + (3 \times 6) + (8 \times 6) = ? \times 6 = ?$

5. $2 \cdot n + 3 \cdot n + 8 \cdot n = ? \cdot n$

6. $7t + 8t = ?t$

7. $5r + 6r = ?$

8. $11s + 5s = ?$

9. $4p + 15p = ?$

10. $10x + 5x + 3x = ?$

11. $2y + 3y + y = ?$

12. $5m + 13m + m = ?$

13. $16A + A + 9A = ?$

14. Similarly, $8 \textit{ times } 3 - 6 \textit{ times } 3 = 2 \textit{ times } 3 = 6$.

15. $(10 \times 5) - (3 \times 5) = ? \times 5 = ?$

16. $(15 \times 9) - (11 \times 9) = ? \times 9 = ?$

17. $11t - 3t = ?t$

18. $16r - 4r = ?$

19. $20y - 11y = ?$

20. $23B - 18B = ?$

21. $36W - 15W = ?$

22. $2x + 6x - 3x = ?$

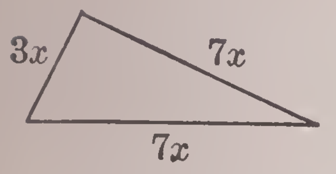
23. $11y - 6y + 9y = ?$

24. $15t + t - 3t = ?$

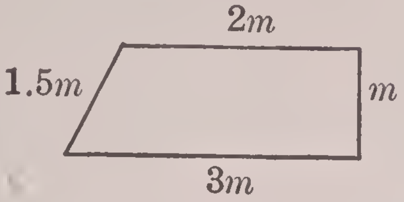
25. $22s + 6s - s = ?$

26. $g + 5g - 3g = ?$

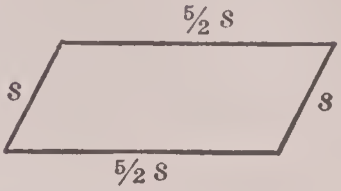
27. What is the perimeter of each of the following figures?



a



b



c

Obtain mentally the results for:

28. $\frac{2}{3}t + \frac{1}{2}t$

29. $\frac{5}{6}r - \frac{1}{2}r$

30. $\frac{3}{5}y - \frac{3}{10}y$

31. $5c + .2c$

32. $3m + .35m$

33. $8.5x - 2.3x$

34. $6.25t - 3.05t$

35. $7\frac{1}{2}w + 2\frac{1}{4}w$

36. $5\frac{1}{2}x - 3\frac{2}{3}x$

37. *a.* What is the perimeter of the square whose sides are m ft. long? *b.* How much is the perimeter when $m = 5$?

38. Answer the same questions if the figure is an equilateral triangle.

39. *a.* The longer side of a parallelogram is 5 times as long as the shorter side. If the shorter side is n inches long, how long is the longer side?

b. What is the perimeter of the rectangle?

40. The length of a certain room is $6x$ ft. and its width is $4x$ ft. Its height is 9 ft. What is the total area of its four walls?

16. Addition and subtraction of literal numbers used in equations.

Example. — Solve the equation $6a + 9a = 50$.

Solution. — 1. $6a + 9a = 50.$
2. Adding, $15a = 50.$
3. D_{15} $a = 3\frac{1}{3}.$

$$50 \div 15 = \frac{\overset{10}{\cancel{50}}}{\underset{3}{\cancel{15}}} = 3\frac{1}{3}$$

Check. — Substituting $3\frac{1}{3}$ for a in the original equation,
Does $6 \times 3\frac{1}{3} + 9 \times 3\frac{1}{3} = 50?$
Does $6 \times \frac{10}{3} + 9 \times \frac{10}{3} = 50?$
Does $20 + 30 = 50?$ Yes.

EXERCISE 12

Solve and check the following equations:

1. $5r + 3r = 96$
2. $11t + 4t = 105$
3. $13x - 4x = 117$
4. $17y - 5y = 180$
5. $2.6z + .9z = 28$
6. $16.5m - 4.2m = 123$
7. $4x + 8x + 6x = 99$
8. $18t - 5t - 2t = 198$
9. $5c + 9c - 6c = 132$
10. $11w - w + 12w = 77$
11. $25z + z - 8z = 96$
12. $s + 15s - 4s = 75$

17. Solving problems by means of equations is the second chief purpose of algebra.

Example. — The sum of two numbers is 96. The larger number is 11 times the smaller. What are the numbers?

Solution. — 1. Let $s =$ the smaller number.
2. Then $11s =$ the larger number.
3. Then $s + 11s =$ the sum of these two numbers.
4. $\therefore s + 11s = 96$, since the sum of the numbers is 96.
5. Adding, $12s = 96$.
6. D_{12} $s = 8$, the smaller number.
7. $\therefore 11s = 88$, the larger number.

Check. — Is the sum of 8 and 88 96? Yes.
Is the larger, 88, eleven times the smaller, 8? Yes.

NOTE. — There are other pairs of numbers whose sum is 96. Thus, 16 and 80. But 80 is not eleven times 16.

Also there are many pairs of numbers such that one is eleven times as large as the other. Thus, 5 and 55. But $5 + 55$ is not 96.

8 and 88 are the only two numbers whose sum is 96 and such that one of them is also 11 times the other.

EXERCISE 13

1. The greater of certain two numbers is 5 times the smaller. The sum of the two numbers is 144. What are the numbers?

2. The sum of certain two numbers is 135. The greater is 14 times the smaller. What are the numbers?

3. James' age is 5 times Fred's age. The sum of their ages is 54 years. How old is each?

4. A man owns two farms having a total of 225 acres. The second farm is 4 times as large as the first. How large is each farm?

5. Two boys made a profit of $3.80 from selling lemonade. They had agreed that the second should receive 3 times as much as the first. How much should each receive?

6. The second of certain three numbers is 3 times the first; the third is 5 times the first. The sum of the three numbers is 126. What are the numbers?

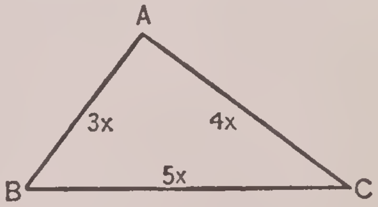
7. The sum of certain three numbers is 480. The second is 4 times the first, and the third is 5 times the first. What are the numbers?

8. The sum of certain three numbers is 260. The second is 3 times the first; the third is 3 times the second. What are the numbers?

9. A man agreed to give a certain amount to a fund provided a second man gave twice as much. A third man agreed to give twice as much as the second provided they made a total contribution of $350. How much did each have to contribute?

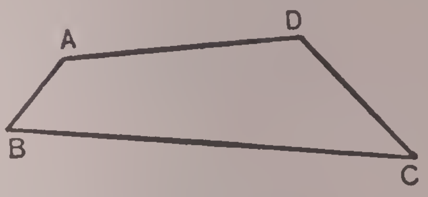
10. The length of a certain rectangle is to be 5 times its width. Its perimeter is to be 360 ft. How long and wide must it be?

11. Find the three sides of the triangle pictured at the right if the perimeter is 540 ft.



12. A boy wants to save $21 in 6 months. He decides to make his savings of the second month twice his savings of the first month; his savings of the third month 3 times his savings of the first month; his savings of the fourth month 4 times his savings of the first month; etc. How much must he save each month?

13. The perimeter of the quadrilateral $ABCD$ is 275 inches. The side CD is twice as long as side AB; side AD is 3 times as long; side BC equals the sum of sides AD and CD. How long is each?

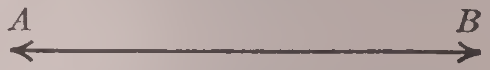


14. The sum of the ages of a father, a mother, and son in a certain family is 60 years. The father is 6 times as old as his son, and the mother is 5 times as old. How old is each?

15. A man had 4500 sq. ft. to devote to flower garden, vegetable garden, and a planting of small fruits. He decided to divide it so that the vegetable garden would be 5 times as large as the flower garden, and the fruit patch 3 times as large as the flower garden. How much space would that give him for each purpose?

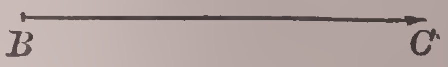
PROBLEMS ABOUT ANGLES

18. A **straight line** extends infinitely far in each of two directions.



This is indicated in the figure above by the arrow heads at the two ends of line AB.

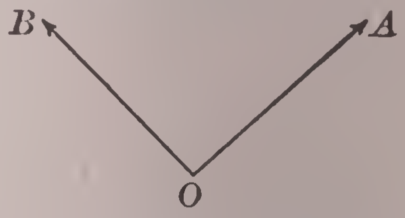
The part of a line on one side of a point, B, is called a **half line** or **ray**. A ray extends infinitely far in one direction. This is indicated by the arrow head at point C.



19. An **angle** is the figure formed by two rays drawn from the same point.

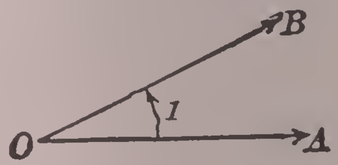
Angle AOB at the right is a right angle.

Find in the room some lines which form a right angle.

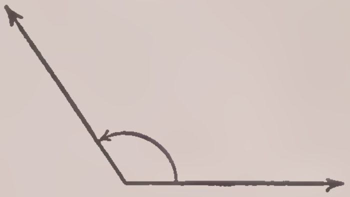


Angles are measured by a small **angle unit** called a **degree.** *A right angle contains 90 degrees.*

An angle smaller than a right angle is called an **acute angle.**



An angle larger than a right angle (and less than two right angles) is called an **obtuse angle.**



EXERCISE 14

1. How many degrees are there in one half a right angle? What kind of angle is this?

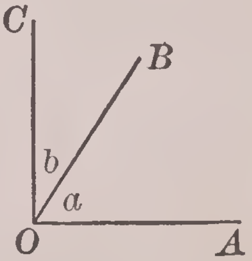
2. How many degrees are there in $\frac{4}{3}$ of a right angle? What kind of angle is this?

3. If an angle measures 30°, how much must you add to it to get a right angle?

4. **Two angles whose sum is a right angle are called complementary angles. Each of the angles is called the complement of the other.**

In the adjoining figure, angles a and b are complementary.

If angle a is 40°, how much is angle b?



5. What is the complement of 60°? 80°? 20°? x°?

6. Are angles of 20° and 65° complementary? Why?

7. If two angles measuring $3x°$ and $5x°$ are complementary, how much is their sum?

Form an equation expressing this fact and, from it, find these two angles.

8. Form an equation expressing the fact that two angles measuring $2x°$ and $7x°$ are complementary, and then find the size of each of these angles.

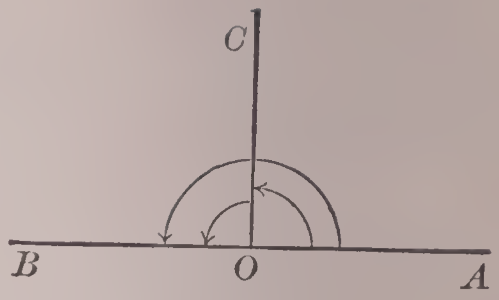
9. *a.* If an angle contains $x°$ and its complement is 9 times as large, what represents its complement?

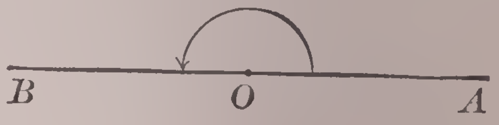
b. Form an equation expressing the fact that these two angles are complementary, and then find the angles.

10. Find the angle which is four times as large as its complement. (Let C = the number of degrees in the complement.)

11. Find the angle which is two thirds of its complement.

12. In the figure at the right how many degrees are there in angle AOC? in angle COB? How many degrees are there in the sum of these two angles?



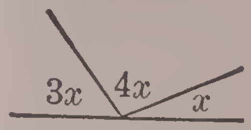


Angle *AOB* is called a straight angle.

A straight angle equals two right angles and measures 180 degrees.

13. Find the size of each angle in the adjoining figure.

14. There are three angles whose sum is 180°. The second is 3 times the first, and the third is 5 times the first. How large is each?



15. *a.* What size of angle must be added to an angle of 60° to make a straight angle?

b. **Two angles whose sum is a straight angle are called supplementary angles. Each of the angles is the supplement of the other.**

16. What is the supplement of 40°? of 90°? of 150°? of $x°$?

17. Form the equation expressing the fact that two angles measuring $4\,x°$ and $8\,x°$ are supplementary. Solve this equation and then determine the size of the angles.

18. *a.* Find the angle which is 8 times as large as its supplement.

b. What kind of angle is each?

19. Find the angle which is 11 times as large as its supplement.

20. *a.* If you examine Figure 1 (p. 25) you will observe that the sum of the four angles is 360°. In Figures 2 and 3, the sum of the four angles must also be 360°.

The sum of the angles around a point is 360°.

b. In Figure 3, since the sum of the angles must be 360°, form an equation expressing this fact, solve it, and find the size of each of the angles.

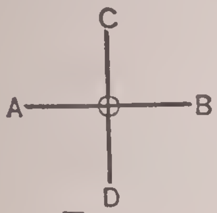


FIG. 1

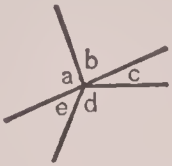


FIG. 2

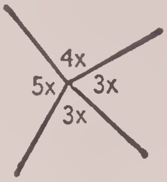
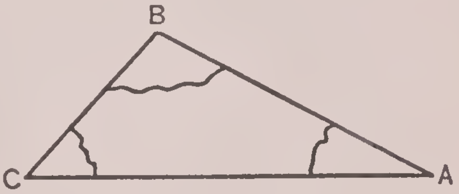
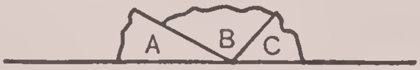


FIG. 3

21. There are four angles making up the total angle around a point. The second angle is 3 times the first; the third is one half the first; the fourth is 3 times the third. How large is each of these angles?

22. *a.* Draw a large triangle on paper, and then cut it from the paper. Tear off the three corners as illustrated at the right. Place these corners side by side as shown in the second figure at the right. You will find that the sum of angles *A*, *B*, and *C* is 180°.





You can prove this fact by measuring the angles, if you know how to use a protractor.

The sum of the angles of a triangle is 180°.

b. If the three angles of a triangle measure $x°$, $2x°$, and $3x°$, write the equation expressing their sum, solve the equation, and find the size of each angle.

23. How large is each of the angles of a triangle if the second is equal to the first and the third is 3 times as large as the first?

24. Find the three angles of a triangle if the second is one third of the first and the third is two thirds of the first.

25. Find the three angles of a triangle if the second and the third are each one half of the first.

26. Find the three angles of a triangle if two angles are equal to each other and the third angle is twice their sum.

EVALUATING ALGEBRAIC EXPRESSIONS

20. An **algebraic expression** is a number expressed by literal and arithmetical numbers, connected by mathematical signs which tell what to do with the numbers.

Thus, $2x - 3yz$ is an expression.

x is to be multiplied by 2; from the result, subtract 3 times the product of z and y.

The arithmetical value of an expression is found by substituting arithmetical values for the literal numbers, and doing what the signs direct.

Thus, if $x = 15$, $y = 2$, and $z = 3$ in $2x - 3yz$,

$$\begin{aligned} \text{then } 2x - 3yz &= 2 \times 15 - 3 \times 2 \times 3 \\ &= 30 - 18 \\ &= 12. \end{aligned}$$

21. Order of fundamental operations. Addition, subtraction, multiplication, and division are called the *four fundamental operations*. Often several of them must be performed when finding the value of an expression, — as in the example above.

Rule. — To evaluate a complicated expression involving the four fundamental operations:

1. First, do all the multiplications.

2. Second, do all the divisions, taking them in order from left to right.

3. Then do the additions and subtractions, taking them in any order.

The example above was solved according to this rule. If, instead, we had first subtracted 3 from 15, getting 12; had multiplied this by 2, getting 24; had multiplied this by 2 and by 3, the final result would have been 144. The correct result is 12.

22. Power, base, and exponent. When the same number is used as a factor several times, the result is called a **power** of the number; the number itself is called the **base**; a small integer written at the right and above the base telling how many times the base is used as a factor is called an **exponent.**

Thus, x^2 means $x \cdot x$. Read it "x square" or "x second power."
x^3 means $x \cdot x \cdot x$. Read it "x cube" or "x third power."
x^4 means $x \cdot x \cdot x \cdot x$. Read it "$x$ fourth" or "x fourth power."

The number x is the base in each of these illustrations; the numbers 2, 3, and 4 are exponents.

EXERCISE 15

Find the numerical value of the following expressions when $a = 2$, $b = 4$, $c = 5$, $d = 3$, $x = 2$, $y = 5$:

1. a^2
2. b^3
3. x^4
4. xy
5. ab^2
6. $2\,cd + xy$
7. $7\,x + 5\,y$
8. $9\,a - 4\,x$
9. $a^3 - x^3$
10. $y^2 - b^2$
11. $a^2 + ab + b^2$
12. $x^2 + 5\,x + 6$
13. $a^3 + b^3 - ab$
14. $(5\,a + 3\,b) \div xy$
15. $(cd - 5\,a) \div bx$

Each of the expressions above can be expressed in words.

Thus, the 13th can be stated:
the cube of a, increased by the cube of b, diminished by the product of a and b.

16–27. Express in words, preferably orally, the algebraic expressions given in Examples 1–12 above.

Write an algebraic expression to represent:

28. The sum of a and b.

29. The product of x and y.

30. The square of x increased by the cube of y.

31. 5 times the product of c and d.

32. The cube of a decreased (made smaller) by the cube of b.

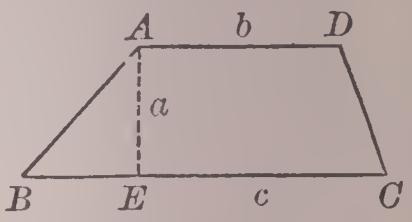
33. The square of x plus 8 times x, diminished by 7.

34. The quotient of a divided by b.

35. The quotient of c divided by d, minus the quotient of x divided by y.

23. Making and evaluating formulæ is one of the useful applications of the previous section.

A rule of computation *expresses* **the relation of one number to one or more other numbers.**



Thus, the area of a trapezoid equals one half its altitude multiplied by the sum of the bases.

This rule gives the relation of the area to the lengths of the altitude and the bases of the trapezoid.

This same rule expressed in algebraic symbols is a formula.

Let a = the number of linear units in the altitude,
b = the number of the same units in the upper base,
c = the number in the lower base,
and A = the number of the corresponding surface units inside the trapezoid;

then

$$A = \tfrac{1}{2}a(b + c).$$

This formula also expresses the relation of the number A to the numbers a, b, and c. Review the manner of using a formula as taught in § 5.

EXERCISE 16

1. Using the formula for the area of a trapezoid, find A when $a = 18$, $b = 20$, and $c = 30$. (See § 5.)

2. Find the area of a trapezoid whose altitude is 8 in., whose upper base is 11 in., and whose lower base is 15 in.

3. In simple interest problems, *the amount equals the principal plus the interest.*

a. Express *this relation* as a formula, using A for the amount, P for the principal, and I for the interest.

b. Using the formula you made in Step *a*, find A when $P =$ \$600 and $I =$ \$16.50.

c. Also find A when $P =$ \$800 and $I =$ \$25.

d. If P remains the same in value, but I increases, what happens to A?

e. If I remains the same in value and P increases, what happens to A?

4. The selling price of an article equals its *cost plus the gain* made when selling it.

a. Express this relation between the selling price, S, the cost, C, and the gain, G.

b. Using your formula, find S when $C = \$6.50$ and $G = \$2.25$.

c. Also find S when $C = 89¢$ and $G = 23¢$.

5. You probably know that *the area of a circle equals pi* (3.1416 *or* $3\frac{1}{7}$) *times the square of the radius.*

a. Express this rule as a formula, using A for the area, π for the number *pi*, and r for the radius.

b. Find, by your formula, the area A when $r = 7$.

c. Find the area of the circle whose diameter is 16 in.

6. It is proved in geometry that *the volume, V, of a sphere equals four thirds of pi times the cube of the radius, r.*

a. Express this relation between V, π, and r as a formula.

b. Using your formula, find V when $r = 3$.

c. Find the volume of the sphere whose radius is 5.

7. Each of n bags of grain weighs k pounds. Let w be their total weight.

a. Express in a formula *the relation between w, n, and k.*

b. By your formula find w when $n = 25$ and $k = 100$.

8. $F = Mv^2/gr$. (Formula from physics.)

Find F when $M = 175$, $v = 25$, $g = 32$, and $r = 5$.

9. $F = \frac{9}{5}C + 32$. (From the study of heat in physics.)

Find F when $C = 90$.

10. $A = P(1 + rt)$. (Formula from the study of simple interest.)

Find A when $P = \$5000$, $r = .06$, and $t = 3\frac{1}{2}$.

11. $S = at + \frac{1}{2}gt^2$. (Formula from mechanics.)
Find S when $a = 500$, $t = 3$, and $g = 32$.

12. $C = nE/(R + nr)$. (From the study of electricity.)
Find C when $E = 1.08$, $n = 40$, $R = 500$, and $r = 4$.

13. $d = i/(1 + i)$. (From the mathematics of investment.)
Find d when $i = .06$.

14. $S = P(1 + r)^2$. (From the study of compound interest.)
Find S when $P = \$2500$ and $r = .05$.

15. $p = ad^2/t + d$. (From an engineer's manual.)
Find p when $a = .56$, $d = \frac{1}{2}$, and $t = \frac{3}{8}$.

Note. — Before starting Chapter II, each pupil should memorize the formulæ collected in § 6, p. 8, and those given in Exercise 16, Examples 1, 3, 5, and 6.

I.e. Example 1. $A = \frac{1}{2}a(b + c)$
Example 3. $A = P + I$
Example 5. $A = \pi r^2$
Example 6. $V = \frac{4}{3}\pi r^3$

Supplementary problems on formulæ appear on page 426.

II. POSITIVE AND NEGATIVE NUMBERS

24. **Another characteristic of algebra is the use in it of numbers called positive and negative numbers.**

These numbers are used to distinguish between opposite quantities.

Thus, rising and falling temperature are opposite quantities. It is not enough to be told that the temperature changed 5 degrees (5°) in a certain time; it is necessary to know whether the temperature rose 5° or fell 5°.

A 5° rise in temperature followed by a 5° fall has a total effect of no change at all.

Similarly, a $50 gain and a $50 loss are opposite quantities. A gain of $50 followed by a loss of $20 has the same effect as a single gain of $30.

EXERCISE 17

What is the opposite of:

1. Sailing 25 miles north?
2. Walking 10 steps forward?
3. Gaining 15 pounds in weight?
4. Depositing $20 in a bank?
5. A 10° fall in temperature?
6. Adding 50?

What is the total effect of:

7. First winning 15 points in a game and then losing 10 points?
8. First walking 12 steps forward and then 8 steps backward?
9. A 10° rise in temperature followed by a 15° fall?

10. Walking 20 steps to the right and then 25 steps to the left?

11. An 8° rise in temperature followed by a 15° fall?

12. Driving an automobile 25 mi. north and then 30 mi. south?

13. First adding 26 and then subtracting 30?

14. First adding 18 and then subtracting 10?

15. First adding 18 and then subtracting 25?

25. The signs + and − are used to distinguish between opposite quantities.

Thus, 8° above zero is written + 8°; and 8° below zero is written − 8°.

Similarly, a $5 gain may be written + $5; and then a $5 loss, − $5.

A quantity preceded by a plus sign is called a positive quantity; one preceded by a minus sign is called a negative quantity.

EXERCISE 18

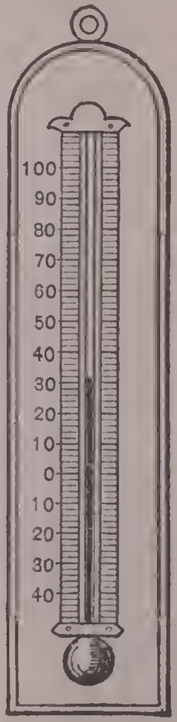
1. The temperature at 8 A.M. was + 10°. At noon, it was 15° warmer. What was the temperature then?

2. *a.* At 7 A.M., the temperature was + 25°. The temperature rose 8° by 10 A.M. What was the temperature then?

b. By 6 P.M., the temperature was 15° lower than it was at 10 A.M. What was the temperature at 6 P.M.?

3. *a.* At 7 A.M., the temperature was − 5°. By noon it had risen 5°. What was the temperature then?

b. By 6 P.M., it had fallen 8° from the temperature at noon. What was it at 6 P.M.?



4. Let us agree to indicate walking to the right by a plus sign, and walking to the left by a minus sign.

What is the total effect of + 10 steps followed by − 15 steps?

Let us agree to write the solution as follows:

(+ 10 steps) + (− 15 steps) = −.5 steps,

but read it as follows:

10 steps to the right, followed by 15 steps to the left, give the same effect as 5 steps to the left.

Similarly, read and tell the result of:

5. (+ 15 steps) + (− 10 steps)

6. (+ 15 steps) + (− 20 steps)

7. (+ 15 steps) + (− 15 steps)

8. (− 5 steps) + (+ 8 steps)

9. (− 8 steps) + (− 2 steps)

10. *a.* If a positive sign indicates a *gain*, what does a negative sign indicate?

b. What is the total result of

(+ \$15) + (+ \$20) + (− \$25)?

Note. — Read this, "a \$15 gain, followed by a \$20 gain, followed by a \$25 loss."

Similarly, read and give the result of:

11. (+ \$20) + (+ \$10) + (− \$15)

12. (+ \$50) + (− \$40) + (+ \$10)

13. (+ \$100) + (− \$25) + (− \$40)

14. (− \$50) + (− \$10) + (− \$15)

15. An automobile was driven + 15 mi. (15 mi. north) from a starting point and then − 8 mi. What was its final distance from the starting point?

16. What was the final distance from its starting point of a taxi which traveled + 3 mi., + 7 mi., − 15 mi., + 4 mi., + 16 mi.?

17. If a boy marks money he receives by a plus sign, and money he spends by a minus sign, how much should he have on hand to agree with the following record:

$$+\$1.35,\quad +25\text{¢},\quad -50\text{¢},\quad +40\text{¢},\quad +35\text{¢},\quad -65\text{¢}\,?$$

18. A grain dealer marked by a plus sign grain he purchased, and, by a minus sign, grain he sold. How much should he have on hand to agree with the following record:

$$+50 \text{ bu.},\quad +75 \text{ bu.},\quad -10 \text{ bu.},\quad -5 \text{ bu.},\quad +25 \text{ bu.},\quad -8 \text{ bu.}\,?$$

19. A teacher marked with a plus sign the number of pupils who enrolled in her room, and by a minus sign the number who dropped out. How many should be left after she made the following record:

$$(+35) + (+2) + (-1) + (-3) + (+1)\,?$$

26. Positive and negative numbers. In algebra, instead of an arithmetical number, like 3, there is the number + 3, called positive 3, and the number − 3, called negative 3.

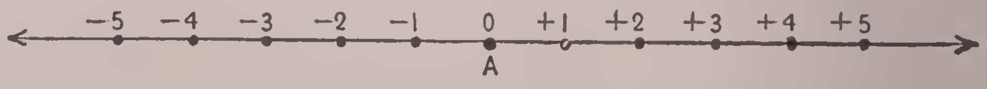
The fundamental fact about the numbers $+3$ and -3 is that

$$(+3) + (-3) = 0.$$

Similarly, $(+\frac{5}{8}) + (-\frac{5}{8}) = 0$; also $(+x) + (-x) = 0$.

Since $(-3) + (+3) = 0$, it is natural to think of -3 as being less than zero.

The following geometrical representation of positive and negative numbers aids in understanding them.



0 is represented on this line by point A.

$+1$, $+2$, $+3$, etc., are represented by points to the right.

-1, -2, -3, etc., are represented by points to the left of the zero point.

Notice that any positive number precedes all larger positive numbers. Since all points marked by negative numbers precede those marked by zero or the positive numbers, every negative number is less than zero and less than any positive number.

Thus, -5 is less than $+1$.

The sign must always be written before a negative number; if no sign appears before a number, the number is considered positive.

The arithmetical number which remains when the sign of a positive or of a negative number is omitted is called the **absolute value of the number.**

Thus, the absolute value of $+5$, or of -5, is 5.

Historical Note. — Hindu mathematicians, who knew about positive and negative numbers long before European mathematicians, in referring to them, used words which correspond to our words for assets and debits. They also were acquainted with the illustration of such numbers by means of the opposite directions on a straight line. To indicate that a number was a negative number they placed a dot over it. European mathematicians did not arrive at an equal understanding of positive and negative numbers until the sixteenth century.

EXERCISE 19

Read the following numbers and tell the absolute value of each.

1. $+15$
2. -60
3. $+\frac{5}{8}$
4. $-\frac{2}{3}$
5. $+.6$
6. $-.08$
7. $+2\frac{3}{4}$
8. -3.07
9. $-\frac{3}{16}$
10. $+.075$

11. *a.* Which is the largest of the numbers in Examples 1–10?
b. Which has the largest absolute value?

12. *a.* Which of the numbers in Examples 1–10 is the smallest?
b. Which has the smallest absolute value?

27. Addition of positive and negative numbers.

Example 1. — Find the sum of $+6$ and $+4$.

Solution. — Just as $6 gain followed by $4 gain gives $10 gain, so

$$(+6) + (+4) = +10.$$

Example 2. — Find the sum of -6 and -4.

Solution. — Just as $6 loss followed by $4 loss gives $10 loss, so

$$(-6) + (-4) = -10.$$

Example 3. — Find the sum of $+6$ and -4.

Solution. — Just as $6 gain followed by $4 loss gives $2 gain, so

$$(+6)+(-4)=+2.$$

Example 4. — Find the sum of -6 and $+4$.

Solution. — Just as $6 loss followed by $4 gain gives $2 loss, so

$$(-6)+(+4)=-2.$$

Rule. — 1. To add two positive numbers, add their absolute value (§ 26) and prefix the plus sign to the result. (Example 1.)

2. To add two negative numbers, add their absolute values and prefix the minus sign to the result. (Example 2.)

3. To add a positive and a negative number,

Subtract the smaller absolute value from the larger and prefix to the result the sign of the number having the larger absolute value. (Examples 3 and 4.)

EXERCISE 20

1. *a.* Add $+3$ to each of the following numbers:

$$+8,\ +7,\ +12,\ +16,\ +20,\ +32.$$

b. Give the rule which you use in this example.

c. What is the sign of the sum when you add positive numbers?

2. *a.* Add -5 to each of the following numbers:

$$-10,\ -7,\ -8,\ -2,\ -20,\ -26.$$

b. Give the rule which you use in this example.

c. What is the sign of the sum when you add two negative numbers?

3. *a.* Add -15 to each of the numbers in Example 1.

b. Give the rule which you use in this example.

4. Add $+16$ to each of the numbers in Example 2.

5. Add $+12$ to each of the following numbers:

$$+15,\ -5,\ +11,\ -3,\ -9,\ +7.$$

6. Add − 9 to each of the numbers in Example 5.

Note. — For further drill, add other numbers to the numbers in Example 5 (*e.g.* + 16, − 5, etc.).

Find the following sums:

7.	**8.**	**9.**	**10.**	**11.**	**12.**
+ 92	− 74	− 85	+ 106	− 240	− 10.5
− 36	+ 60	− 37	− 72	+ 155	− 6.4

13.	**14.**	**15.**	**16.**	**17.**	**18.**
− 6	+ 15	− 24	+ 68	− 84	+ 9.5
+ 22	− 35	+ 40	− 96	+ 107	− 15.6

19. Find
+ 5
− 4
+ 7
− 9

Hint. — Either add in order from the bottom to the top; or, first add the positive numbers, obtaining + 12; then, add the negative numbers, obtaining − 13; then add these two results.

20.	**21.**	**22.**	**23.**	**24.**	**25.**
+ 5	− 6	+ 12	− 6	+ 14	− 8
− 7	− 4	− 10	− 11	− 12	+ 11
− 3	+ 8	+ 5	+ 10	− 9	− 12
+ 4	− 2	+ 7	+ 5	+ 20	+ 9
− 5	+ 7	− 8	− 7	− 7	− 4

Find the sum of:

26. $-\frac{1}{2}$ and $+\frac{3}{4}$ **28.** $-\frac{5}{8}$ and $-\frac{1}{2}$ **30.** $+2\frac{1}{4}$ and $-3\frac{1}{2}$

27. $+\frac{5}{6}$ and $-\frac{1}{3}$ **29.** $+\frac{3}{5}$ and $-\frac{7}{10}$ **31.** $-3\frac{1}{6}$ and $+2\frac{5}{6}$

28. Adding a negative number gives the same result as subtracting the corresponding arithmetical number.

Thus, 12 + (− 3) gives the same result as 12 − 3. Each equals + 9.

29. Multiplication of positive and negative numbers.

The words **multiplier, multiplicand,** and **product** have the same meaning in algebra as in arithmetic.

Rules for multiplication of signed numbers are suggested by the following examples.

The sign × is to be read "multiplied by."

Example 1. — Find $(+4)$ multiplied by $(+3)$.

Solution. — $(+4) \times (+3) = (+4) + (+4) + (+4) = +12.$

$$\therefore\ (+\mathbf{4}) \times (+\mathbf{3}) = +\mathbf{12}.$$

Example 2. — Find (-4) multiplied by $(+3)$.

Solution. — $(-4) \times (+3) = (-4) + (-4) + (-4) = -12.$

$$\therefore\ (-\mathbf{4}) \times (+\mathbf{3}) = -\mathbf{12}.$$

Example 3. — Find $(+4)$ multiplied by (-3).

Solution. — 1. In arithmetic, $4 \times 3 = 3 \times 4$; that is, the multiplier and multiplicand may be interchanged.

2. If it is assumed that the same law holds for negative numbers, then

$$(+4) \times (-3) \text{ should equal } (-3) \times (+4).$$

3. But $(-3) \times (+4) = (-3) + (-3) + (-3) + (-3) = -12.$

4. $\therefore$ also $(+\mathbf{4}) \times (-\mathbf{3}) = -\mathbf{12}.$

Note. — When multiplying by a negative number, -3, it appears that the result, -12, may be obtained by multiplying first as if the multiplier were positive, $+3$, getting $+12$, and then changing the $+12$ to its opposite, -12.

Example 4. — Find (-4) multiplied by (-3).

Solution. — 1. Using the idea presented in the note following Example 3, first multiply (-4) by $(+3)$. This gives -12. (See Example 2.)

2. Now take the opposite of -12. This gives $+12$.

3. $(-\mathbf{4}) \times (-\mathbf{3}) = +\mathbf{12}.$

Rule. — To multiply one signed number by another:

1. Find the product of their absolute values. (See all four examples above.)

2. Make the product positive if the multiplicand and multiplier have like signs. (See Examples 1 and 4 above.)

3. Make the product negative if the multiplicand and multiplier have unlike signs. (See Examples 2 and 3 above.)

EXERCISE 21

1. Multiply the following numbers by + 5:

$$+3,\ +7,\ +9,\ +6,\ +11,\ +\tfrac{3}{5},\ +\tfrac{5}{6}.$$

2. Multiply the numbers in Example 1 by − 3.

3. Multiply the following numbers by + 4:

$$-7,\ -10,\ -5,\ -8\ -12,\ -\tfrac{2}{3},\ -\tfrac{4}{5}.$$

4. Multiply the numbers in Example 3 by − 5.

5. Multiply the numbers in Example 1 by − 6.

6. Multiply the numbers in Example 3 by + 10.

7. Multiply the numbers in Example 1 by − 4.

8. Multiply the numbers in Example 3 by − 6.

9. What is the sign of the product when the multiplier and multiplicand have the same sign? when they have unlike signs?

10. Multiply each of the following numbers by − 8:

$$+6,\ -10,\ +12,\ -9,\ -4,\ -\tfrac{1}{2},\ +\tfrac{3}{4}.$$

11. Multiply the numbers in Example 10 by + 12.

12. Multiply the numbers in Example 10 by + 16.

13. Multiply the numbers in Example 10 by − 20.

14. Multiply the following numbers by − 24:

$$-\tfrac{1}{2},\ +\tfrac{1}{3},\ -\tfrac{1}{4},\ -\tfrac{5}{6},\ -\tfrac{2}{3},\ -\tfrac{7}{12}.$$

15. Multiply the numbers in Example 14 by + 36.

Find the following products:

16. $(+.7) \times (-1.5)$
17. $(-6) \times (+.25)$
18. $(-.4) \times (-.8)$
19. $(+2.5) \times (-1.5)$
20. $(-3.2) \times (-1.2)$
21. $(-3) \times (+2) \times (-4)$
22. $(+5) \times (-5) \times (+5)$
23. $(-6) \times (-2) \times (-3)$
24. $(-1) \times (+4) \times (-2)$
25. $(+7) \times (-5) \times (+2)$

30. **Powers of positive and negative numbers.**

$$(+2)^2 = (+2)\cdot(+2) = +4$$
$$(+2)^3 = (+2)\cdot(+2)\cdot(+2) = +8$$
$$(+2)^4 = (+2)\cdot(+2)\cdot(+2)\cdot(+2) = +16$$

Clearly, every power of a positive number is positive.

$$(-2)^2 = (-2)\cdot(-2) = +4$$
$$(-2)^3 = (-2)\cdot(-2)\cdot(-2) = -8$$
$$(-2)^4 = (-2)\cdot(-2)\cdot(-2)\cdot(-2) = +16$$
$$(-2)^5 = (-2)\cdot(-2)\cdot(-2)\cdot(-2)\cdot(-2) = -32$$

31. Clearly, every even power of a negative number is positive, and every odd power is negative.

EXERCISE 22

Find the values of the following:

1. $(-3)^2$
2. $(-3)^4$
3. $(-3)^3$
4. $(-3)^5$
5. $(-4)^2$
6. $(-4)^3$
7. $(-2)^6$
8. $(-5)^3$
9. $(-5)^4$
10. $(+1)^6$
11. $(-1)^5$
12. $(-1)^6$

13. Find the value of $ax^2 + bx + c$ when $a = 5$, $b = 2$, $c = -3$, and $x = -2$.

Solution. — 1. $ax^2 + bx + c = 5\cdot(-2)^2 + 2(-2) + (-3)$
2. $= 5\cdot(+4) + (-4) + (-3)$
3. $= 20 - 4 - 3$
4. $= 13.$

NOTE. — Observe that $+(-4)$ gives -4; and $+(-3)$ gives -3, according to § 28, page 37.

Find the values of the following expressions, when $a = 3$, $b = -2$, $c = -3$, and $x = -2$. (Read again § 21, page 26.)

14. $5\,ac$
15. a^2b
16. bc^2
17. $3\,bx$
18. $2\,x^2 + 3\,x$
19. $ax^2 - bx$
20. $bx^2 + cx^2$
21. $ax^3 - bx^3$
22. $3\,x^2 + 2\,x + 1$
23. $ax^2 + bx + c$
24. $x^2 + 2\,ax + a^2$
25. $b^2 - 2\,bc + c^2$

NOTE. — Attention is called to the tests on pages 453–454.

III. ADDITION AND SUBTRACTION OF POLYNOMIALS

32. A **monomial** or **term** consists of numbers connected only by signs of multiplication or division. Thus,

$2x^2$, $-3ab$, and $+5$ are the terms of the expression $2x^2 - 3ab + 5$.

If the sign before a term is plus, as $2x^2$, it is called a **positive term**; if minus, as $-3ab$, it is called a **negative term.**

33. If two or more numbers are multiplied together, each of them or the product of any number of them is a **factor** of the product.

Thus, a, b, c, ab, ac, and bc are all factors of abc.

34. Any factor of a product is called the **coefficient** of the product of the remaining factors.

Thus, in the term $2ab$, 2 is the coefficient of ab, $2a$ of b, and $2b$ of a.

35. In a term like $5x^2y$, the factor 5 is called the **numerical coefficient** of x^2y.

A term like x has the numerical coefficient 1, since x is the same as $1x$.

If a term is negative, like $-3a$, the numerical coefficient is -3, since $(-3) \cdot a = -3a$.

36. Like terms are terms which have the same literal factors.

Thus, $2x^2y$ and $-5x^2y$ are like terms; also, $2\left(\frac{x}{y}\right)$ and $3\left(\frac{x}{y}\right)$.

Unlike terms are terms which do not have the same literal factors.

Thus, $2x^2y$ and $2xy^2$ are unlike terms.

Unlike terms may be like with respect to one or more letters.

Thus, $2axy$ and $3byx$ are like with respect to xy.

EXERCISE 23

1. Give five different factors of $8\,xy$.
2. What is the coefficient of xy in $8\,xy$? of y?
3. What is the numerical coefficient of ab in $-6\,ab$?
4. Are $8\,x^3$ and $-7\,x^3$ like terms?
5. Are $8\,x^2$ and $-7\,x^3$ like terms?

37. Addition of like terms with positive coefficients has been taught previously.

Thus, $6\,n + 4\,n = 10\,n$.

n is the common factor of $6\,n$ and $4\,n$. 6 and 4 are the coefficients of n. Their sum, 10, is the coefficient of the result.

Rule. — To add two or more like terms, multiply their common factor by the sum of its coefficients.

This rule is employed also when the coefficients are negative.

Example. — What is the sum of $-5\,x^2y$ and $+3\,x^2y$?

Solution. — 1. The coefficients of x^2y are -5 and $+3$, and the sum of these coefficients is -2.

2. $$\therefore\ (-5\,x^2y) + (+3\,x^2y) = -2\,x^2y.$$

No matter what the values of x and y may be, the sum of $(-5\,x^2y)$ and $(+3\,x^2y)$ is always $-2\,x^2y$.

Thus, if $x = 2$ and $y = 1$

$$-5\,x^2y = -5\cdot 4\cdot 1 = -20;\ 3\,x^2y = 3\cdot 4\cdot 1 = 12;$$

and $$(-20) + (+12) = -8.$$

Also, $$-2\cdot x^2\cdot y = -2\cdot 4\cdot 1 = -8.$$

For this reason, the correctness of addition of polynomials can be checked by substituting numbers.

EXERCISE 24

Find the sum of:

1. $8\,A$ and $-15\,A$
2. $9\,p$ and $-12\,p$
3. $-11\,m$ and $+8\,m$
4. $-15\,t$ and $-4\,t$
5. $16\,E$ and $-27\,E$
6. $-17\,r^2s$ and $+26\,r^2s$
7. $-xyz$ and $+14\,xyz$
8. $-bc^2$ and $-7\,bc^2$

Time yourself to see how quickly you can get the sums in the following examples:

9.	**10.**	**11.**	**12.**	**13.**
$-18p$	$+17x^2y$	$-5rs$	$+9w^2$	$-28xyz$
$+11p$	$-4x^2y$	$-16rs$	$-36w^2$	$+42xyz$

14.	**15.**	**16.**	**17.**	**18.**
$-5t$	$+17x^2$	$+9ab$	$+xyz$	$7ab^2c$
$+12t$	$-3x^2$	$-10ab$	$-4xyz$	$-ab^2c$
$-9t$	$+8x^2$	$-13ab$	$+6xyz$	$-8ab^2c$
$-4t$	$-12x^2$	$+7ab$	$-13xyz$	$+14ab^2c$
$+6t$	$+5x^2$	$+5ab$	$+5xyz$	$-9ab^2c$

19.	**20.**	**21.**	**22.**	**23.**
$-\frac{1}{4}mn$	$-\frac{2}{3}rs$	$-\frac{3}{5}x^3$	$-2.5ab$	$-.25y^2$
$+\frac{1}{2}mn$	$+\frac{1}{2}rs$	$+\frac{1}{2}x^3$	$+3.4ab$	$-3.5y^2$
$-mn$	$-\frac{1}{6}rs$	$-\frac{4}{10}x^3$	$-.6ab$	$+4.5y^2$

38. Adding unlike terms. If asked to add 5 gal. and 3 qt., you can write 5 gal. + 3 qt. This expression indicates the sum. You get the actual sum only when you change 5 gal. to 20 qt. Then, 5 gal. + 3 qt. = 20 qt. + 3 qt. = 23 qt.

Similarly, if asked to add $5x$ and $3y$, **you can only indicate the sum** by writing $5x + 3y$.

You cannot combine $5x$ and $3y$ in any way since they are unlike terms.

Later, if you learn that $x = 4$ and $y = 6$, you can find the value of the sum. Then $5x + 3y = 20 + 18 = 38$.

Addition of unlike terms can only be indicated, and that is done by writing them with a plus sign between them.

39. Recall that adding a negative number has the same effect as subtracting the corresponding positive number.

Thus, $10 + (-4) = 10 - 4;$

and $5x + (-2x) = 5x - 2x.$

Example. — *a.* Write in simplest form the sum of $2x^2$, $-3x$, -4, $-5x^2$, $+7$, and $+8x$.

Solution. — 1. Indicating the addition, the sum is

$$(2x^2) + (-3x) + (-4) + (-5x^2) + (+7) + (+8x).$$

2. And this is the same as,

$$2x^2 - 3x - 4 - 5x^2 + 7 + 8x. \qquad (\S\ 28)$$

3. Combining like terms, the sum is $-3x^2 + 5x + 3$.

b. What is the value of the sum, when $x = 2$?

Solution. — 1. $-3x^2 + 5x + 3 = (-3 \cdot 4) + 5 \cdot 2 + 3$

2. $= -12 + 10 + 3 = +1.$

EXERCISE 25

Indicate the sum of the terms in each of the following examples, and simplify the result by combining like terms:

1. $7ab$, $-3ab$, $+4ab$
2. $9r^2$, $-7r^2$, $+4r^2$, $-8r^2$
3. $12xy$, $-10xy$, $+6xy$
4. $-4z$, $+7z$, $-10z$
5. $16t^2$, $-11t^2$, $-12t^2$
6. $-\frac{1}{2}xy$, $\frac{3}{4}xy$, $-\frac{3}{8}xy$
7. $\frac{2}{3}m^2$, $-\frac{1}{6}m^2$, $-\frac{5}{12}m^2$
8. $-\frac{3}{5}w$, $\frac{1}{4}w$, $+\frac{9}{10}w$

9. *a.* $8x^2$, $-9x$, $-6x^2$, $+5$, $+4x$, -7
b. What is the value of the sum, when $x = 2$?

10. *a.* $4a$, $-5b$, $-3c$, $+7b$, $+c$, $-2a$
b. What is the value of the sum, when $a = 3$, $b = 2$, $c = 1$?

11. $10m^3$, $+6n^2$, $-8mn$, $-4m^3$, $+5mn$, and $-2n^2$

12. $10r^2$, $-6r$, $+5s$, $-8r$, $+2s$, $-5r^2$, and $-4s$

13. $9r^3$, $-6r^3$, $+3r^2$, $-7r$, $+8r^2$, $-15r$, and $+5$

14. $2.5xy$, $-1.2xy$, $+.5xy$, and $-3.8xy$

15. $2.4a^3$, $-5a^2$, $+.7a^3$, -3, $+2a^2$, and -1

40. A **polynomial** is an algebraic expression of two or more terms, as, $a + b$ or $x^2 - 2xy + 5y^2$.

A **binomial** is a polynomial having two terms; as $a + b$.

A **trinomial** is a polynomial having three terms; as $x^2 - 2xy + 5y^2$.

NOTE. — If a number expression is inclosed in parentheses, it is considered a term.

Thus, $a + (b + c)$ is, for the time, a binomial having the term a and the term $(b + c)$.

41. Arrangement of terms in ascending and descending powers of a letter.

If the terms of a polynomial are $-3x^2y$, $+x^3$, $-y^3$, and $+3xy^2$, a natural orderly arrangement of them is as follows:

$$x^3 - 3x^2y + 3xy^2 - y^3.$$

These terms are arranged in descending powers of x.

The exponents of x in order are now 3, 2, 1, and 0, because $-y^3$ does not have any factor x.

The terms are arranged in ascending powers of y.

The exponents of y in order are 0, 1, 2, 3, for the first term, x^3, does not contain any factor y.

42. Addition of polynomials is like addition of denominate numbers in arithmetic.

Example 1. — Add 3 yd. 2 ft. 6 in. and 5 yd. 4 in.

Solution.

$$\begin{array}{lll} 3\text{ yd.} & 2\text{ ft.} & 6\text{ in.} \\ \underline{5\text{ yd.}} & \underline{\phantom{2\text{ ft.}}} & \underline{4\text{ in.}} \\ 8\text{ yd.} & 2\text{ ft.} & 10\text{ in.} \end{array}$$

In this solution, like units are placed in the same column.

Example 2. — Add $4a + 5b - 2c$ and $-3a + 2b - 6c$.

Solution.

$$\begin{array}{r} 4a + 5b - 2c \\ \underline{-3a + 2b - 6c} \\ a + 7b - 8c \end{array}$$

In this example, a, b, and c may represent any numbers.

Thus, if $a = 2$, $b = 3$, and $c = 1$,

$$\left.\begin{aligned} 4a + 5b - 2c &= 8 + 15 - 2 = 21 \\ -3a + 2b - 6c &= -6 + 6 - 6 = -6 \end{aligned}\right\} 21 + (-6) = 15.$$

Also, the sum $a + 7b - 8c = 2 + 21 - 8 = 15$.

That is, the value of $4a + 5b - 2c$, added to the value of $-3a + 2b - 6c$, is the same as the value of the sum, $a + 7b - 8c$.

This fact is most important. It can be used when checking a solution.

Rule. — To add polynomials:

1. Rewrite the polynomials, if necessary, in descending powers, or ascending powers of one letter, with like terms in the same column.

2. Add the columns of like terms and combine the results.

3. Check the solution by adding again in the opposite direction, or by substituting values for the letters.

Example 3. — Find the sum of $x^3 - y^3 + 3xy^2 - 3x^2y$, $2x^3 + 3xy^2 + y^3$, and $2x^2y - 4xy^2$. Check, by letting $x = 2$ and $y = 1$.

Solution. — 1. Arrange the terms in descending powers of x, with like terms in the same column.

$$\begin{array}{rrrrcl}
x^3 & - 3x^2y & + 3xy^2 & - y^3 & = & 8 - 12 + 6 - 1 = 1 \\
2x^3 & & + 3xy^2 & + y^3 & = & 16 \qquad + 6 + 1 = 23 \\
 & + 2x^2y & - 4xy^2 & & = & \qquad 8 - 8 \qquad = \underline{0} \\
\hline
3x^3 & - x^2y & + 2xy^2 & & & \qquad\qquad\qquad\quad 24
\end{array}$$

2. Also, $3x^3 - x^2y + 2xy^2 = 24 - 4 + 4 = 24$.

3. $\therefore$ The solution is correct.

EXERCISE 26

Find the following sums, checking by substitution:

1.

$$\begin{array}{rrr}
+ 7x^2 & - 2x & + 1 \\
- 2x^2 & + 4x & - 3 \\
- 3x^2 & + x & + 2 \\
\hline
\end{array}$$

2.

$$\begin{array}{rr}
- 10A & + 5B \\
+ 4A & - 3B \\
+ 2A & - 6B \\
\hline
\end{array}$$

3.

$$\begin{array}{rrr}
+ 3m & - 4n & + 7p \\
- 5m & + 9n & - 6p \\
+ 7m & - 6n & - 5p \\
\hline
\end{array}$$

4.

$$\begin{array}{rrr}
a^2 & - 3a & + 1 \\
a^2 & + 9a & - 6 \\
2a^2 & & + 3 \\
\hline
\end{array}$$

5.

$$\begin{array}{rrr}
x^2 & & + y^2 \\
2x^2 & + xy & \\
 & - 3xy & - 2y^2 \\
\hline
\end{array}$$

6.

$$\begin{array}{rrr}
m^2 & + 2mn & + n^2 \\
2m^2 & & - 3n^2 \\
 & - 3mn & + 2n^2 \\
\hline
\end{array}$$

NOTE. — Observe that there are blank spaces in Examples 4, 5, and 6 for powers which are missing from the polynomials.

Arrange the following examples as in Examples 1–6 and find the sums. *Check each by adding in the opposite order.* Time yourself to see how quickly you can get all of them.

7. $6a - 5b + 7c$ and $4a + 6b - 4c$

8. $5k - 7l + 3m$, $4k + 3l - 5m$, and $8k - 2l$

9. $3a + 9b - 6c$, $-4b + 3c - 6d$, and $7c - 3a + 4d$

10. $x^2 - 2xy - y^2$, $x^2 + 2xy - y^2$, and $2x^2 + 2y^2 - 5xy$

11. $5m^2 - 3mn + n^2$, $m^2 - 4mn - 4n^2$, and $-3m^2 + 6n^2$

12. $a^3 + 3a^2b + 3ab^2 + b^3$ and $a^3 - 3a^2b - b^3 + 3ab^2$

13. $3x^2 - 8xy + 4y^2$, $2xy - 4x^2 - 7y^2$, and $6y^2 + 3xy + 5x^2$

14. $15a^3 - 7 - 3a + 16a^2$ and $+4 + 9a - 20a^2 - 10a^3$

15. $-4x + 2x^3 - 5$, $6x^2 - 3x + 6x^3$, and $13 - 2x^3 + 9x - 8x^2$

16. $2a^2 - 3b^2$, $-3ab + 8b^2 - 5a^2$, and $9a^2 + 6ab - 10b^2$

17. $\frac{1}{2}a - \frac{1}{3}b + \frac{1}{5}c$ and $-\frac{1}{4}a + \frac{1}{6}b - \frac{1}{10}c$

18. $\frac{2}{3}m + \frac{3}{5}n - r$, $-\frac{1}{3}m - \frac{7}{10}n + \frac{1}{2}r$, and $m + \frac{3}{4}r - \frac{1}{2}n$

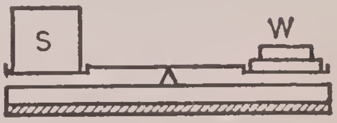
19. $2.5A - 6.25B + 4.8C$ and $-1.8A + 5B - 3.4C$

20. $2.25x^2 - 3.4xy - 2y^2$ and $-1.75x^2 + 2.1xy - 1.5y^2$

NOTE. — Supplementary examples on addition of polynomials are to be found on page 427.

43. Addition used in solving equations.

In the adjoining figure, suppose the sugar S just balances the weight W. If a 5 lb. weight is added to the right scale pan and 5 lb. of sugar to the left pan, then the two pans will still balance. Also the two pans will not balance if a 5 lb. weight is added to the right pan and 3 lb. or anything other than 5 lb. to the left.



Similarly, if $3x = 60$, then $3x + 5 = 60 + 5$.

Rule. — The same number can be added to both sides of an equation without destroying the equality.

Example 1. — Solve the equation $x - 5 = 13$.

Solution. — 1. $x - 5$ is 5 less than x. If 5 is added to $x - 5$, the result is $x - 5 + 5$ or x.

But, if 5 is added to the left side of the equation, it must also be added to the right side in order to keep the two sides equal.

Therefore, add 5 to both sides of the equation.

2. Then, $x - 5 + 5 = 13 + 5$. (Rule, page 47)

3. Combining like terms, $x = 18$.

Check. — Substituting 18 for x in the *original equation*,
does $18 - 5 = 13$? Yes.

Then, 18 is the correct root of the equation.

NOTE. — For meaning of "root of the equation," see § 10, page 10.

Hereafter, instead of writing in full
"*add 5 to both sides of the previous equation*" we shall write A_5, and instead of "combining like terms" we shall write C. T.

Example 2. — Solve the equation $5y - 13 = 22$.

Solution. — 1. $5y - 13 = 22$.
2. A_{13} $5y = 35$.
3. D_5 $y = 7$.

$5y - 13 + 13 = 5y$
$22 + 13 = 35$
(This should be done mentally.)

Check. — Substitute 7 for y in the original equation.
Does $5 \cdot 7 - 13 = 22$?
Does $35 - 13 = 22$?
Yes. Therefore 7 is the correct root of the equation.

EXERCISE 27

Solve and check the following equations:

1. $x - 10 = 5$
2. $y - 12 = 16$
3. $2z - 5 = 13$
4. $3t - 7 = 14$
5. $4n - 9 = 27$
6. $m - 14 = -8$
7. $A - 16 = -3$
8. $2B - 21 = -5$
9. $5t - 28 = -8$
10. $6r - 40 = -2$

11. Solve the equation $7x = 55 - 4x$.

Solution. — 1. $7x = 55 - 4x$.

We must remove $4x$ from the right side.

2. A_{4x} $11x = 55$.

$7x + 4x = 11x$

$55 - 4x + 4x = 55$

(Complete the solution and check it.)

NOTE. — What does A_{4x} in Step 2 mean?

12. $5x = 32 - 3x$

13. $7t = 27 - 2t$

14. $3y = 42 - 4y$

15. $8w = 130 - 5w$

16. $16A = 13 - 10A$

17. $25v = 8 - 15v$

18. Solve the equation $3x - 42 = -4x$.

Solution. — 1. $3x - 42 = -4x$.

We must remove -42 from the left side.

2. A_{42} $3x = 42 - 4x$.

$3x - 42 + 42 = 3x$

$-4x + 42 = 42 - 4x$

(Complete the solution.)

19. $5y - 70 = -2y$

20. $7m - 80 = -3m$

21. $12s - 95 = -7s$

22. $18x - 24 = -6x$

23. $9z - 4 = -3z$

24. $20r - 24 = -12r$

25. $3c - 3 = 13 - c$

26. $4d - 12 = -3 - 5d$

27. $-16 + 2x = 9 - 3x$

28. $7w - 18 = 6 - w$

29. $13s - 11 = 4 - 2s$

30. $5y - 8 = -5y - 3$

SUBTRACTION OF SIGNED NUMBERS AND EXPRESSIONS

44. Meaning of subtraction.

To subtract 4 from 7 means to determine the number which must be added to 4 to give 7.

To subtract a from b means to determine the number which must be added to a to give b.

This meaning of subtraction is used in "making change."

The number subtracted is called the **subtrahend.**

The number from which the subtrahend is subtracted is called the **minuend.**

The result is called the **difference** or **remainder.**

EXERCISE 28

What number must be added to the first number of each of the following examples to get the second?

1. 3, 10
2. 8, 15
3. 10, 19
4. $3a$, $5a$
5. $4m$, $8m$
6. $9t$, $17t$
7. $14c$, $26c$
8. $2b$, $23b$
9. $5x$, $19x$

How much must the temperature rise to change:

10. $+4^\circ$ into $+9^\circ$?
11. -6° into -2°?
12. -3° into $+5^\circ$?
13. -5° into $+8^\circ$?

Find the number to be added:

14. $(+4) + ? = +10$
15. $(-5) + ? = 0$
16. $(-4) + ? = +2$
17. $(-6) + ? = +3$
18. $(-10) + ? = +2$
19. $(-6) + ? = -2$
20. $(-7) + ? = +8$
21. $(-15) + ? = -10$

How much must the temperature fall to change:

22. $+10^\circ$ to $+2^\circ$?
23. $+10^\circ$ to 0°?
24. $+10^\circ$ to -5°?
25. -4° to -12°?

Find the number to be added:

26. $(+5) + ? = +3$
27. $(+5) + ? = 0$
28. $(+5) + ? = -3$
29. $(+10) + ? = 0$
30. $(+10) + ? = -5$
31. $(-2) + ? = -10$
32. $(-4) + ? = -15$
33. $(+3) + ? = -6$
34. $(+10) + ? = -4$
35. $(+15) + ? = -10$
36. $(-6) + ? = +5$
37. $(+5) + ? = -4$
38. $(-7) + ? = -16$
39. $(-10) + ? = -4$
40. $(+3) + ? = -11$
41. $(+7) + ? = -1$
42. $(-9) + ? = -20$
43. $(-6) + ? = +13$
44. $(+2) + ? = +20$
45. $(-2) + ? = -20$

45. Rule for subtracting signed numbers.

The rule for subtraction is taught by these four problems and their solutions in columns *A* and *B*	(*A*) Sign of the Subtrahend as Given. Subtract	(*B*) Sign of the Subtrahend Changed. Add
Example 1. — Subtract + 2 from + 6. This means (+ 2) + ? = + 6. The result is + 4 (Column *A*). This same result is obtained in Column *B* by changing + 2 to − 2 and adding.	$\begin{array}{r} +6 \\ +2 \\ \hline +4 \end{array}$	$\begin{array}{r} +6 \\ -2 \\ \hline +4 \end{array}$
Example 2. — Subtract − 2 from + 6. This means (− 2) + ? = + 6. The result is + 8 (Column *A*). This same result is obtained in Column *B* by changing − 2 to + 2 and adding.	$\begin{array}{r} +6 \\ -2 \\ \hline +8 \end{array}$	$\begin{array}{r} +6 \\ +2 \\ \hline +8 \end{array}$
Example 3. — Subtract + 2 from − 6. This means (+ 2) + ? = − 6. The result is − 8 (Column *A*). The same result is obtained in Column *B* by changing + 2 to − 2 and adding.	$\begin{array}{r} -6 \\ +2 \\ \hline -8 \end{array}$	$\begin{array}{r} -6 \\ -2 \\ \hline -8 \end{array}$
Example 4. — Subtract − 2 from − 6. This means (− 2) + ? = − 6. The result is − 4 (Column *A*). The same result is obtained in Column *B* by changing − 2 to + 2 and adding.	$\begin{array}{r} -6 \\ -2 \\ \hline -4 \end{array}$	$\begin{array}{r} -6 \\ +2 \\ \hline -4 \end{array}$

Since the correct results are obtained in Column *B*, we have the following

Rule. — To subtract one number from another, change the sign of the subtrahend and add the resulting number to the minuend.

Example. — Subtract -6 from -13.

Solution. — Mentally change -6 to $+6$; then add $+6$ and -13. The result is -7.

$$\begin{array}{r} -13 \\ \underline{-\ \ 6} \\ -\ \ 7 \end{array}$$

Check. — The sum of -6 and -7 is -13, as it should be.

Historical Note. — The symbol $-$, like the symbol $+$, first appeared in print in a mathematical book by Widmann in 1489. The Italian and French mathematicians of the same period used the symbol m̃, derived from the first letter of the Latin word *minus*.

EXERCISE 29

Subtract the lower number from the upper number in each of the following examples:

$$\begin{array}{cccccccc} \textbf{1.} & \textbf{2.} & \textbf{3.} & \textbf{4.} & \textbf{5.} & \textbf{6.} & \textbf{7.} & \textbf{8.} \\ +16 & +20 & +12 & +9 & -15 & -13 & -5 & -\ 8 \\ \underline{+\ 7} & \underline{+13} & \underline{-\ 4} & \underline{-5} & \underline{+\ 8} & \underline{+\ 4} & \underline{+7} & \underline{+10} \end{array}$$

$$\begin{array}{cccccccc} \textbf{9.} & \textbf{10.} & \textbf{11.} & \textbf{12.} & \textbf{13.} & \textbf{14.} & \textbf{15.} & \textbf{16.} \\ +8a & -7x & +15y^2 & -13t & -19m & +18A & -20x^2 & +14rs \\ \underline{-3a} & \underline{+9x} & \underline{-\ 7y^2} & \underline{-\ 5t} & \underline{-\ 6m} & \underline{-\ 5A} & \underline{+\ 3x^2} & \underline{-\ 8rs} \end{array}$$

17. *a.* Was it possible to subtract 15 from 12 in arithmetic?

b. What is the result in algebra when you subtract 15 from 12?

18. *a.* Which is the subtrahend in a subtraction example? Which is the minuend?

b. Is it necessary to write the subtrahend below the minuend?

19. *a.* Subtract -5 from 13.

b. Subtract 13 from -5 without rewriting the numbers.

$$\begin{array}{r} 13 \\ \underline{-\ 5} \end{array}$$

20–27. In Examples 9–16, subtract the upper numbers from the lower without rewriting the numbers.

28. From $-15a^3$ take $-23a^3$.

29. Subtract $18abc$ from $3abc$.

30. From $-5m^2$ subtract $-8m^2$.

46. Subtraction of polynomials.

Rule. — To subtract one polynomial from another:

1. Rewrite the minuend, if necessary, in descending powers of one letter.

2. Below it, write the subtrahend, so that like terms are in the same column.

3. Imagine the signs of the terms of the subtrahend changed, and add the resulting terms to those of the minuend.

Example. — Subtract $7ab^2 - 9a^2b + 8b^3$ from $-2a^2b + 4ab^2 + 5a^3$.

Solution. — 1. Arrange the terms in descending powers of a, with like terms in the same column.

$$\begin{array}{ll} \text{Minuend} & 5a^3 - 2a^2b + 4ab^2 \\ \text{Subtrahend} & \underline{ - 9a^2b + 7ab^2 + 8b^3} \\ \text{Change signs mentally, and add} & 5a^3 + 7a^2b - 3ab^2 - 8b^3 \end{array}$$

NOTE. — In the first column, since nothing is added to $5a^3$, the result is $5a^3$.

In the fourth column, $+8b^3$ becomes $-8b^3$. Since nothing is added to it, the result is $-8b^3$.

Check. — The sum of the remainder and subtrahend is the minuend, as it should be.

EXERCISE 30

1.

$$\begin{array}{r} 5a - 8b + 11c \\ \underline{2a + 3b - 5c} \end{array}$$

2.

$$\begin{array}{r} 11x + 13y - 14z \\ \underline{4x + 9y - 16z} \end{array}$$

3.

$$\begin{array}{r} 2r + 7s - 10t \\ \underline{2r - 4s + 2t} \end{array}$$

4.

$$\begin{array}{r} 12a^2 - 10a - 6 \\ \underline{9a^2 - 6a + 5} \end{array}$$

5.

$$\begin{array}{r} 4x^2 + 13x - 10 \\ \underline{-3x^2 - x - 5} \end{array}$$

6.

$$\begin{array}{r} 3x^2 - 5xy + y^2 \\ \underline{2x^2 - 3xy + 5y^2} \end{array}$$

7.

$$\begin{array}{rrr} 12\,r^2 & & -\,6\,s^2 \\ 9\,r^2 & -\,4\,rs & +\,2\,s^2 \\ \hline \end{array}$$

8.

$$\begin{array}{rrrr} a^3 & -\,3\,a^2b & & -\,b^3 \\ a^3 & +\,3\,a^2b & +\,3\,ab^2 & +\,b^3 \\ \hline \end{array}$$

9.

$$\begin{array}{rrr} x^4 & +\,x^2y^2 & +\,y^4 \\ x^4 & -\,x^2y^2 & +\,y^4 \\ \hline \end{array}$$

10.

$$\begin{array}{rrrr} x^3 & -\,5\,x^2 & +\,6\,x & \\ & 2\,x^2 & & -\,5 \\ \hline \end{array}$$

11.

$$\begin{array}{rrrr} y^3 & & +\,3\,y & -\,2 \\ & -\,y^2 & +\,4\,y & -\,1 \\ \hline \end{array}$$

12.

$$\begin{array}{rrrr} a^3 & -\,3\,a^2 & & +\,4 \\ & +\;\;a^2 & -\,2\,a & -\,2 \\ \hline \end{array}$$

Subtract:

13. $5\,a - 6\,b + 7\,c$ from $13\,a - 4\,b + 8\,c$

14. $2\,x^2 - 3\,xy + 4\,y^2$ from $3\,x^2 - 5\,xy - 6\,y^2$

15. $x^2 + 4\,xy - 3\,y^2$ from $2\,x^2 - 5\,xy + y^2$

16. $4\,x^3 - 6\,x^2 + 8\,x - 9$ from $7\,x^3 - 16\,x + 18$

17. $ax - bx + cx$ from $2\,ax - bx + cx$

18. $-3\,z + 11\,x - 4\,y$ from $15\,x - 5\,y + 8\,z$

19. $6\,A - 5\,B + 3\,C$ from $8\,A + 2\,B - 7\,C$

20. $2\,x^3 - 7 + 4\,x - 5\,x^2$ from $6\,x^2 - 8 + 10\,x^3 - 3\,x$

21. From $1 + a^3 - 2\,a + a^2$ take $3\,a - 2 + a^2 - a^3$

22. Subtract $3\,x^3 - 5 + 4\,x - 2\,x^2$ from $7\,x^3 - 5\,x + 12 - 5\,x^2$

23. Subtract $2\,x^3 - 3\,x^2 + 4\,x - 5$ from 0

24. Subtract $\frac{1}{2}\,m - \frac{1}{4}\,n + \frac{1}{5}\,p$ from $\frac{2}{3}\,m - \frac{1}{8}\,n + \frac{1}{4}\,p$

25. Subtract $.5\,x - .3\,y + 1.4\,z$ from $2.8\,x - .5\,y + 2.5\,z$

26. From the sum of $3\,a^2 - ab - 2\,b^2$ and $2\,a^2 + 5\,ab - 3\,b^2$ subtract $a^2 - 3\,ab - 4\,b^2$

27. From $6\,x^2 - 7\,xy + 8\,y^2$ subtract the sum of $x^2 - 3\,xy - y^2$ and $2\,x^2 + 5\,xy - 4\,y^2$

28. From the sum of $a^2 - 2\,ab + b^2$ and $2\,a^2 + 2\,ab + b^2$ take the sum of $a^2 - b^2$ and $a^2 + ab + 3\,b^2$

29. How much greater than $m^2 + 3\,mn - n^2$ is $4\,m^2 - 5\,mn + 6\,n^2$?

30. *a*. How much less than $3x^2 - 7x + 9$ is $2x^2 + 4x - 8$?
b. What is the value of the result when $x = 2$?

NOTE. — Additional drill examples of the same kind are to be found in Exercise 218 on page 428, and some of a more difficult kind in Exercise 219 on page 429.

47. Subtraction used in solving equations.

After reading again the rules in § 11, § 12, and § 43, you easily understand the

Rule. — The same number can be subtracted from both members of an equation without destroying the equality.

Example 1. — Solve the equation $2x + 17 = 33$.

Solution. — 1. We can remove 17 from the left side of this equation by subtracting 17 from it. We must then subtract 17 from the other side *also* to keep the two sides equal.

		$2x + 17 - 17 = 2x$
2. S_{17}	$2x = 16$	$33 - 17 = 16$
3. D_2	$x = 8$	This can be done mentally.

Check. — Substitute 8 for x *in the original equation.*
Does $2 \cdot 8 + 17 = 33$? Does $16 + 17 = 33$?
Yes. Therefore 8 is the correct root of the equation.

NOTE. — In Step 2, S_{17} means subtract 17 from both sides of the preceding equation.

Example 2. — Solve the equation $5t + 6 = 2t + 27$.

Solution. — 1. $5t + 6 = 2t + 27$.
2. S_6 $5t = 2t + 21$.
3. S_{2t} $3t = 21$.
4. D_3 $t = 7$.

Check. — Substitute 7 for t *in the original equation.*
Does $5 \cdot 7 + 6 = 2 \cdot 7 + 27$? Does $35 + 6 = 14 + 27$?
Yes, since $35 + 6 = 41$, and $14 + 27 = 41$.
Therefore 7 is the correct root of the equation.

NOTE. — In Step 2, $5t + 6 - 6 = 5t$; $2t + 27 - 6 = 2t + 21$.
In Step 3, $5t - 2t = 3t$; $2t + 21 - 2t = 21$.
All this is to be done mentally.

EXERCISE 31

Solve and check the following equations:

1. $x + 13 = 25$
2. $3y + 11 = 32$
3. $5t + 7 = 52$
4. $4k = 60 + k$
5. $7B = 28 + 3B$
6. $9r = 55 + 4r$
7. $12z = 84 + 5z$
8. $10x + 5 = 13 + 6x$
9. $13y + 3 = 19 + 8y$
10. $7s + 8 = 14 + 6s$
11. $15m + 7 = 7m + 55$
12. $10t + 9 = 4t + 12$
13. $20w + 11 = 15w + 12$
14. $18x + 5 = 15x + 7$

Solve the following equations, using the rules taught in § 43 and § 47:

15. $18y - 7 = 3y + 23$
16. $3s - 3 = 2 + 2s$
17. $4y + 2 = 3y + 8$
18. $3t + 9 = 37 - t$
19. $5w + 1 = 29 - 2w$
20. $16r - 13 = 4r + 23$
21. $12x + 5 = 12 - 2x$
22. $15b - 4 = 3 - 6b$
23. $15b + 4 = 11 - 6b$
24. $32W - 6 = 1 + 4W$
25. $10x - 1 = 6x + .6$
26. $8y + 2 = 3y + 5.5$
27. $7r + 6 = 8.2 - 4r$
28. $4c + 6 = 12 - c$
29. $17t - 4 = 8 - 7t$
30. $13p + 5 = -2p + 10$

48. Forming algebraic expressions for number relations.

In arithmetic, you can actually find the sum of two numbers like 5 and 8. **In algebra, you can only represent the sum of 5 and x by writing $5 + x$.**

EXERCISE 32

1. Represent 8 more than y; 10 less than y.

2. *a.* How much is 10 increased by 3?
b. Represent 10 increased by n; also x increased by 15.

3. *a.* How much is 20 decreased by 9? 24 decreased by 18?
b. Represent 15 decreased by x; by m.
c. Represent y decreased by 4; by 7.

4. *a.* Represent m increased by 6.

b. What is the value of the expression found in part *a*, when m is 4? 7? 10?

5. A man spends y dollars out of his income of $200.

a. How much has he left?

b. What is the value of the expression in part *a*, when y is 25? 50? 100?

6. *a.* How much does 12 exceed 9? 22 exceed 15?

b. How much does $5x$ exceed $3x$? exceed 10?

c. How much does y exceed 14? 20 exceed x?

7. The altitude of a certain rectangle is h inches. Its base is 4 in. longer.

a. Represent its base.

b. Represent its perimeter.

8. One number is 4 times as large as a certain smaller number. The smaller number is represented by s.

a. Represent the larger number.

b. Represent 15 more than the smaller number.

c. Suppose the results of parts *a* and *b* are equal. Give the equation expressing this fact.

9. One number is 6 times as large as a certain smaller number. Let x represent the smaller number.

a. Represent the larger number.

b. Represent 20 more than the smaller number.

c. Represent 5 less than the larger number.

d. Suppose the results of parts *b* and *c* are equal. Give the equation expressing this fact.

10. One number is 13 less than a certain larger number. Let g represent the larger number.

a. Represent the smaller number.

b. Represent the sum of the two numbers.

c. Suppose that the sum of the two numbers (represented in part *b*) is 88. Give the equation expressing this fact.

Form equations for the following by direct translation, but do not solve the equations:

Thus, A number decreased by 13 = 76
Write $n \quad - \quad 13 = 76$

11. Three times a certain number equals 15 diminished by the number.

12. Twice a certain number increased by 5 equals 18 diminished by three times the number.

13. If 9 be added to four times a certain number, the result equals 15 more than 3 times the number.

14. Four fifths of a certain number increased by 16 equals the given number less 10.

15. If 9 be added to 7 times a certain number, the result equals 16 diminished by the number.

49. Solving problems by means of equations.

A problem asks for values of one or more unknown numbers.

Rule. — 1. Represent one unknown number by a letter.

2. Represent the other unknown numbers of the problem by means of this same letter, using relations of these unknowns to the first unknown.

3. Form an equation expressing an equality which is stated in the problem.

a. Sometimes this equation expresses the fact that certain two results are equal.

b. Sometimes this equation expresses the fact that some expression containing the unknown number has a definite numerical relation to some known number.

4. Solve the equation, thus getting the unknown number represented by the letter. Then find all the other unknown numbers required.

Example. — One number is 9 times as large as another. If 16 be subtracted from the larger, the result is 32 more than the smaller. What are the numbers?

Solution. — 1. Let $s =$ the smaller number.
2. $\therefore 9s =$ the larger number.
3. $9s - 16$ is 16 less than the larger number.
4. $s + 32 =$ 32 more than the smaller number.
5. $\therefore 9s - 16 = s + 32$, because the problem states these two results are equal.
6. A_{16} $9s = s + 48$.
7. S_s $8s = 48$.
8. D_8 $s = 6$, the smaller number.
9. $\therefore 9s = 54$, the larger number.

Check. — If 16 be subtracted from 54, the result is 38. If 32 be added to 6, the result is 38. Since these two results are equal, and since 54 is nine times 6, then 54 and 6 must be the correct numbers.

NOTE. — *Always check by going back to the statements made in the problem.*

EXERCISE 33

1. What number increased by 45 equals 63?

2. One number is 9 times as large as a certain smaller number. If the smaller number is subtracted from the larger, the remainder is 32. Find the number.

3. The result obtained by subtracting 24 from 7 times a certain number is 25. What is the number?

4. If 12 be subtracted from 9 times a certain number, the result equals 7 times the number. What is the number?

5. There are two numbers of which the larger is 7 times the smaller. Also the larger is 48 more than the smaller. What are the numbers?

6. One of two line segments is 4 times as long as the other. If 20 in. be added to the shorter, and 4 in. be taken from the longer, the two results will be equal. How long are the segments?

7. If 13 times a certain number be increased by 5, the result is 40 more than 8 times the number. What is the number?

8. If 3 be subtracted from 32 times a certain number, the result equals 15 more than 20 times the number. What is the number?

9. The length of certain lots in one city is 5 times their width. The perimeter of these lots is 300 ft. What are the length and the width of the lots?

10. The base of a certain rectangle is 3 times as long as its altitude. If the perimeter be increased by 10, the result is 50. What are the base and altitude of the rectangle?

11. The length of a certain parallelogram is 3 times its width. If the perimeter is increased by 6 ft., the result will be 110 ft. What are the length and width of the parallelogram?

12. 8 times a certain number exceeds (is more than) 3 times the number by 95. What is the number?

13. 11 times a certain number exceeds 14 by 63. What is the number?

14. If 10 times a certain number be diminished by 15, the result is the same as if 6 times the number were increased by 21. What is the number?

15. Fifteen times a certain number diminished by 8 gives the same result as 6 times the number diminished by 5. What is the number?

16. Eight times a certain number exceeds 37 by the same amount that 23 exceeds twice the number. What is the number?

17. The longest river in the world is the combined Mississippi-Missouri, and the next longest is the Nile. The length of the former is 700 mi. more than the length of the latter. Together, their length is 7700 mi. How long is each?

18. The area of the Caspian Sea, the largest salt-water lake, is 180,000 sq. mi. This area is 20,000 sq. mi. more than 5 times the area of Lake Superior, the largest fresh-water lake. What is the area of Lake Superior?

19. *a.* An article cost C dollars. Represent 15 % of this cost.

b. Represent the cost plus 15 % of the cost.

c. Suppose the result of part *b* is $2.99. Form the equation expressing this fact.

d. Solve the equation formed in part *c*, and thus determine the cost of the article.

20. What is the cost of an article which sells for \$7.44 if the gain is 24 % of the cost?

50. Most practical problems requiring an algebraic solution are solved by using a formula.

The formula for the amount in simple interest problems is $A = P + PRT$ because the amount equals the principal plus the interest, and the interest is PRT. In this formula, T is a number of years, and R is the rate of interest per year.

Example. — At what rate of simple interest will \$1500 amount to \$2000 in 5 years?

Solution. — 1. The formula is $A = P + PRT$.
$A = 2000;\ P = 1500;\ R = ?\quad T = 5.$
2. Substituting $2000 = 1500 + 1500 \cdot R \times 5.$
3. $\therefore\ 2000 = 1500 + 7500\,R.$
4. S_{1500} $500 = 7500\,R$ or
$7500\,R = 500.$
5. D_{7500} $R = \frac{1}{15}$, or $6\frac{2}{3}\%$.

Check. — The interest on \$1500 at $6\frac{2}{3}\%$ for 1 yr. = \$100.
$\therefore$ the interest for 5 yr. = \$500.
$\therefore$ the amount = \$1500 + \$500, or \$2000.
$\therefore$ $6\frac{2}{3}\%$ is the correct rate.

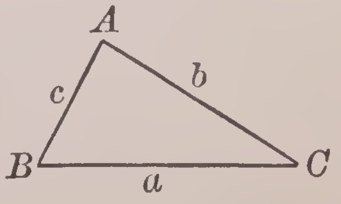
NOTE. — Use the above form of solution in the following examples.

EXERCISE 34

1. If P represents the perimeter of a triangle whose sides have lengths a, b, and c, then

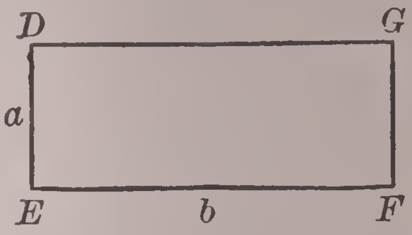
$$P = a + b + c$$

is the formula for the perimeter.



a. Find P when $a = 15$, $b = 40$, $c = 27$.
b. Find a when $P = 67$, $b = 20$, $c = 30$.
c. Find b when $P = 96$, $a = 22$, $c = 43$.
d. Find c when $P = 72$, $a = 32.5$, $b = 16.8$.

2. The formula for the perimeter of a rectangle whose altitude is a and base is b is



$$P = 2a + 2b.$$

a. Find P when $a = 18$ and $b = 45$.
b. Find a when $P = 300$ and $b = 40$.
c. Find b when $P = 275$ and $a = 46$.

3. $l = a + (n - 1)d$ is an important formula in mathematics.

a. Find l when $a = 5$, $n = 23$, and $d = 2$.
b. Find a when $l = 68$, $n = 19$, and $d = 3$.
c. Find d when $l = 275$, $a = 25$, and $n = 26$.

4. Using the formula for the amount in simple interest problems:

a. Find P when $A = \$3500$, $R = 5\%$, and $T = 8$ yr.
b. Find R when $A = \$5000$, $P = \$4000$, and $T = 2\frac{1}{2}$ yr.
c. Find T when $A = \$2600$, $P = \$2000$, and $R = 6\%$.

5. By the formula $S = \frac{na}{2} + \frac{nl}{2}$:

a. Find S when $n = 10$, $a = 13$, and $l = 63$.
b. Find l when $S = 400$, $a = 5$, and $n = 40$.
c. Find a when $S = 600$, $l = 32$, and $n = 30$.

NOTE. — Additional problems of this sort are on page 430. Also attention is called to the tests on page 455.

IV. PARENTHESES

51. Parentheses are used to group together terms which are to be treated as a single number expression.

Thus, $$3a - (2x + y - z)$$
means that $2x + y - z$ is to be subtracted from $3a$.

Brackets, [ ], **braces,** { }, and the **vinculum,** $\overline{}$, are used in the same manner. For convenience, they are referred to collectively as parentheses, or as *symbols of grouping.*

HISTORICAL NOTE. — Parentheses were introduced by Girard, a Dutch mathematician, about 1629. Previously, the brackets and the braces had been used by Vieta, about 1593, and by Bombelli, an Italian, about 1572. Descartes used the vinculum.

52. Removing parentheses.

Example 1. — Find the value of $x^2 + (-x + 5)$.

Solution. — $x^2 + (-x + 5)$ means that $-x + 5$ is to be added to x^2.

$$\begin{array}{lr} & x^2 \\ & -x+5 \\ \cline{2-2} \text{Adding,} & x^2 - x + 5 \end{array} \qquad \begin{array}{l} \therefore\ x^2 + (-x+5) \\ = x^2 - x + 5. \end{array}$$

NOTE. — The parentheses do not appear in the result; *they have been removed.* The sign of x in the result is $-$, just as it was inside the parentheses; also the sign of 5 is unchanged.

Rule. — Parentheses preceded by a plus sign may be removed without changing the signs of the terms which are within the parentheses.

Example 2. — Find the value of $x^2 - (-x + 5)$.

Solution. — $x^2 - (-x + 5)$ means that $-x + 5$ is to be subtracted from x^2.

$$\begin{array}{lr} & x^2 \\ & -x+5 \\ \cline{2-2} \text{Subtracting,} & x^2 + x - 5 \end{array} \qquad \begin{array}{l} \therefore\ x^2 - (-x+5) \\ = x^2 + x - 5. \end{array}$$

NOTE. — *The parentheses have been removed.* The sign of x in the result is $+$, whereas it was $-$ inside the parentheses; the sign of 5 in the result is $-$, whereas it was $+$ inside the parentheses.

Rule. — Parentheses preceded by a minus sign may be removed provided the signs of all terms which are within the parentheses are changed from + to −, or from − to +.

Example 3. — Remove the parentheses and collect like terms in $2a - 6b + (4a - b) - (5a - 4b)$.

Solution. — 1. $4a$ and $-b$, inclosed by parentheses preceded by a plus sign, do not have their signs changed when the parentheses are removed.

$5a$, which means $+5a$, and $-4b$, inclosed in parentheses preceded by a minus sign, must have their signs changed.

2. $$\therefore 2a - 6b + (4a - b) - (5a - 4b) = 2a - 6b + 4a - b - 5a + 4b.$$

3. Collecting like terms, the result is $a - 3b$.

EXERCISE 35

Remove symbols of grouping and collect like terms:

1. $3m - (m + 4)$

2. $4x + (2 - 3x)$

3. $6y - (5 - 4y)$

4. $7t - (-3t - 2)$

5. $8m + \{-2m - 5\}$

6. $7a + (-2a + b)$

7. $6r - [-s + 3r]$

8. $8c + \{-3c + 2b\}$

9. $9k - [3k - 1]$

10. $10m - \{-3n + 5m\}$

Ex. No.	Remove Parentheses from	Find the Value of the Result
11.	$3m + (5m - 2)$	when $m = 2$
12.	$7x - (4x + 3)$	when $x = 5$
13.	$5t - (6 - t)$	when $t = 2$
14.	$2y + (-7 + y)$	when $y = 3$
15.	$3z - (-6 + z)$	when $z = 4$

In the following ten examples, remove both sets of parentheses at the same time. Combine like terms.

16. $(4c + 3d) - (2c - d)$
17. $(x - 2y) - (x + 3y)$
18. $(2r + s) - (r - 4s)$
19. $[r + s] - [t + v]$
20. $(x - y) - (2y - x)$
21. $\{2a - 3b\} - \{5a - 6b\}$
22. $[-a + b] - [c - d]$
23. $(-a - b) + (-c - d)$
24. $(a^2 - a) - (a^2 + a)$
25. $(5x - 8) - (3x + 4)$

53. Removing parentheses which are within other parentheses.

Example.— Simplify $6x - (3x + \{2x - 4\} - \{x - 5\})$.

Solution. — The braces inclosing $2x - 4$ and $x - 5$ are removed first, giving attention only to the sign + before $\{2x - 4\}$ and the sign − before $\{x - 5\}$. The rest of the expression in merely rewritten.

1. $6x - (3x + \{2x - 4\} - \{x - 5\})$
2. Removing { } $= 6x - (3x + 2x - 4 - x + 5)$
3. C. T. inside ( ) $= 6x - (4x + 1)$
4. Removing ( ) $= 6x - 4x - 1$
5. C. T. $= 2x - 1.$

Note. — If $x = 5$, $2x - 1 = 9$. Therefore the original expression also has the value 9 when $x = 5$.

Rule. — To simplify an expression having parentheses within parentheses:

1. Remove the innermost parentheses first, according to the rules in § 52.

2. Combine the terms within the resulting innermost parentheses. Then remove these parentheses.

3. Continue until all parentheses are removed and like terms are combined.

EXERCISE 36

1. $8r + [5r + (2s - t)]$
2. $3x + [7x - \{2x + 1\}]$
3. $9y + (2y - [8y - 4])$
4. $11z + (-4z + [2z - 5])$
5. $6r + [-15 - (3r - 9)]$
6. $12 - \{6x + (4 - 5x)\}$
7. $9t - \{7 + (3t - 2)\}$
8. $4m - [3m - (2m + 1)]$

9. $6w - \{4 - (5w - 8)\}$

10. $5A - [-2A - (3A - 1)]$

11. $(7t - 5) - (3t - \{2t + 1\})$

12. $(3m - 4) - (-2m + [5m - 6])$

13. $(11w - 5) - (-4 - [6w - 3])$

14. $9r + \{3r - (6r + 1) + (2r - 8)\}$

15. $7x + (8 - [5x - 2] + [-3x + 4])$

16. $6b - [4b - \{2 + b\} - \{3b + 1\}]$

17. $(4t - 5) - (3 + \{2t - 7\} - \{6 + 5t\})$

18. $4x - \{5x - [6x + 7a - 2] + 5a\}$

19. $2m - [(3m - 4) - (5m + 6)]$

20. $8c - \{5c - [2c - 6] + [3c - 7]\}$

NOTE. — Supplementary examples are to be found on page 430.

54. Placing terms inside parentheses.

Development. — 1. What is the rule for removing parentheses preceded by a plus sign?

2. What, then, should be done with the signs of terms which are placed within parentheses preceded by a plus sign?

Example 1. — Place the last two terms of $a + b - c + d$ in parentheses preceded by a plus sign.

Solution. — $a + b - c + d = a + b + (-c + d)$.

Check. — When the parentheses are removed

$a + b + (-c + d)$ becomes $a + b - c + d$.

3. What is the rule for removing parentheses preceded by a minus sign?

4. What, then, should be done with the signs of terms which are placed within parentheses preceded by a minus sign?

Example 2. — Inclose the last two terms of $a - b - c + d$ in parentheses preceded by a minus sign.

Solution. — The signs of $-c$ and $+d$ must be changed.

$\therefore\ a - b - c + d = a - b - (+c - d)$.

Check. — $a - b - (c - d) = a - b - c + d$ when the parentheses are removed.

Rule. — 1. If terms are placed within parentheses preceded by a plus sign, their signs must not be changed.

2. If terms are placed within parentheses preceded by a minus sign, their signs must be changed.

EXERCISE 37

Inclose the last three terms of the following expressions in parentheses preceded by a plus sign:

1. $x + y + z - w$

2. $r + s - t + v$

3. $m - n + p - g$

4. $a - b - c - d$

5. $x^2 + x - 5 - y$

6. $a^2 - b^2 - 2bc - c^2$

7. $x^2 + y^2 - 2yz + z^2$

8. $4m^2 - n^2 - 6n - 9$

9. $x^2 - 4ab - a^2 - 4b^2$

10. $m - nx - px - q$

11–20. Inclose the last three terms of the expressions in Examples 1–10 in parentheses preceded by a minus sign.

21–30. Inclose the last two terms of the expressions in Examples 1–10 in parentheses preceded by a minus sign.

55. Using parentheses in equations.

Example. — Solve and check the equation

$$(8t - 11) - (7 - 5t) = 12 - (13 + 4t).$$

Solution. — 1. $(8t - 11) - (7 - 5t) = 12 - (13 + 4t).$

2. Removing ( ) $8t - 11 - 7 + 5t = 12 - 13 - 4t.$

3. C. T. $13t - 18 = -1 - 4t.$

4. A_{18} $13t = 17 - 4t.$

5. A_{4t} $17t = 17.$

6. D_{17} $t = 1.$

Check. — Substituting 1 for *t in the original equation,*

does $(8 - 11) - (7 - 5) = 12 - (13 + 4)$?

Does $-3 - 2 = 12 - 17$? Does $-5 = -5$? Yes.

$\therefore$ 1 is the correct root of the equation.

EXERCISE 38

Solve and check the following equations:

1. $3x - (x - 4) = 14$
2. $(3y - 4) - (7 + y) = 7$
3. $6z + (7z - 11) = 15z - (4z + 5)$
4. $(4m - 13) - (5 + 2m) = (1 - 4m) - (30 - 7m)$
5. $(10t + 3) - (8t - 7) = (15 - 6t) - (14t - 6)$
6. $(3A - 4) - (2A - 10) = (15 - 8A) - (1 - 5A)$
7. $(6W - 5) - (9W - 4) = (6 - 12W) - (15W - 1)$
8. $(17 - 2s) - (3s - 15) + (4s - 13) = 3s - 21$
9. $(5r - 4) - (11 - 15r) = (25r - 3) - (10r + 7)$
10. $(4c + 7) - (5 - 8c) = 20c - (12c - 5)$

56. Using parentheses to represent number relations.

Example. — Indicate and then simplify the excess of $7x$ over the binomial $(3x - 4)$.

Solution. — 1. "Excess" means "amount more than."

$\therefore$ the excess of 12 over 9 is $12 - 9$ or 3.

2. Thus, the excess of $7x$ over $(3x - 4) = 7x - (3x - 4)$.

3. Removing ( ) the excess $= 7x - 3x + 4$, or $4x + 4$.

EXERCISE 39

Indicate and then simplify, if possible:

1. The sum of x and $(x + 7)$; of x and $(x - 5)$.
2. $(2x + 5)$ decreased by $(4 - x)$.
3. $(3m - 8)$ decreased by $(2m + 5)$.
4. The remainder when $(2x - 3)$ is subtracted from $(5x + 6)$.
5. The result when $(3 + y)$ is diminished by $(y - 9)$.
6. The amount by which $(5r - 4)$ exceeds $(12r + 3)$.
7. The excess of 18 over x; of 18 over $(x - 5)$.
8. The excess of $(3z + 4)$ over $(2z - 8)$.

9. The smaller part of 18 if the larger part is l.

10. The larger part of 20 if $(x + 3)$ is the smaller part.

11. The number n is separated into two parts. If 10 is the larger part, what is the smaller?

12. A line segment of length 25 in. is separated into two parts, of which the shorter is s. Represent the longer part.

13. *a.* 5, 6, and 7 are three consecutive integers. What are the two integers consecutive to them?

b. What must you do to an integer to get the consecutive integer?

c. If x is an integer, what is the first consecutive integer? The next?

14. *a.* 2, 4, and 6 are consecutive even integers. What are the two even integers consecutive to them?

b. If x represents an even integer, what are the two consecutive even integers?

15. *a.* 7, 9, 11, 13 are four consecutive odd integers. What are the two odd integers consecutive to them?

b. If x is an odd integer, what are the two consecutive odd integers?

c. What are these three integers if x is 21?

d. What are these three integers if x is -5?

16. *a.* Let $x =$ an integer. Then represent the two consecutive integers.

b. Now represent the sum of the three integers obtained in part *a.*

17. Let a equal the altitude of a certain rectangle.

a. If the base is 4 in. more than the altitude, represent the base.

b. Represent the perimeter of this rectangle.

c. How much is this perimeter when $a = 5$?

18. The shortest side of a certain triangle is x in. long.

a. Represent the second side if it is 8 in. longer than the first side.

b. Represent the third side if it is 3 in. less than twice the first side.

c. Represent the perimeter of this triangle.

57. Using parentheses to solve problems.

Example. — Separate 140 into three numbers such that the second is 15 less than the first and the third exceeds twice the first by 35.

Solution. — 1. Three numbers are to be found. Their sum is 140. Each must be represented.

2. Let $f =$ the first number.
3. $\therefore f - 15 =$ the second, since it is 15 less than the first.
4. And $2f + 35 =$ the third, since it exceeds twice the first by 35.
5. $\therefore f + (f - 15) + (2f + 35) = 140$, since the sum of the numbers is 140.
6. Removing ( ) $f + f - 15 + 2f + 35 = 140.$
7. C. T. $4f + 20 = 140.$
8. S_{20} $4f = 120.$
9. D_4 $f = 30$, the first number.
10. $\therefore f - 15 = 15$, the second number.
11. $\therefore 2f + 35 = 95$, the third number.

Check. — 1. The second is 15 less than the first, and the third exceeds twice the first by 35.

2. Does $30 + 15 + 95 = 140$? Yes. $\therefore$ 30, 15, and 95 are the required numbers.

NOTE 1. — Use parentheses as in Step 5.

NOTE 2. — Read Step 5 thus:

f plus the binomial $f - 15$ plus the binomial $2f + 35$ equals 140.

NOTE 3. — Read again the suggestions given in § 49, p. 58.

EXERCISE 40

1. One number is 7 more than a smaller number. The sum of the numbers is 35. What are the numbers?

2. One number exceeds another number by 13. Their sum is 25. What are the numbers?

3. Separate 100 into two parts such that the larger is 14 more than the smaller. What are the numbers?

4. Separate 124 into two parts such that the greater exceeds the smaller by 56.

5. A boy has a board 36 in. long. He wants to cut it into two pieces such that one will be 8 in. longer than the other. How long will the pieces be?

6. Two angles, *A* and *B*, are complementary. (See Example 4, p. 23.) Angle *A* is 21° smaller than angle *B*. How large is each?

7. (Review Example 22, p. 25.) Let angles *A*, *B*, and *C* be the three angles of a triangle. If angle *A* is 50° less than angle *B*, and angle *C* is 5° more than angle *B*, how large is each?

8. The sum of the ages of three children of a certain family is 30 years. Charles' age is 1 yr. more than twice John's age; Mary's age is 1 yr. less than 3 times John's age. What are their ages?

9. The base of a certain rectangle is 5 in. longer than the altitude. The perimeter of the rectangle is 66 in. Find the base and altitude.

10. The perimeter of a certain triangle is 44 in. The second side is 1 in. less than twice the first side; the third side is 5 in. more than twice the first side. How long is each side?

11. The total height of the Library of Congress at Washington is 64 ft. The height of its second story is 7 ft. less than twice the height of its first story; the height of its third story is 1 ft. more than twice the height of its first story. What is the height of each of its three stories?

12. There are two consecutive integers whose sum is 49. What are they? (Review Examples 13 and 16, page 69.)

13. There are three consecutive integers whose sum is 126. What are they?

14. There are two consecutive odd integers whose sum is 144. What are they? (Review Example 15, page 69.)

15. (Review Example 15, p. 24.) Certain two angles are supplementary. One exceeds the other by 45°. How large is each?

16. Separate 75.6 into two parts such that the larger is 49 more than the smaller.

17. Separate 55.5 into two parts such that the larger exceeds the smaller by 17.9.

18. A board 14 ft. long is to be cut into two pieces such that one will be $1\frac{1}{2}$ ft. longer than the other. How long will each piece be?

19. Divide $175 between three children so that the first will receive $10 less than the second, and the third will receive $20 more than the second.

20. Charles' age is 2 yr. more than twice John's age. Henry's age is 1 yr. more than three times John's age. The sum of their ages is 33 years. How old is each?

21. The base of a certain rectangle is 4 in. less than 5 times the altitude. The perimeter of the rectangle is 1 in. more than 9 times the altitude. What are the base and altitude?

22. The sum of certain three consecutive integers is 204. What are they?

23. If the first of certain three consecutive even integers is increased by the second, the sum is 20 more than the third integer. What are the integers?

24. Eight major planets, including the earth, revolve around the sun. The number of planets which are farther from the sun than the earth is one less than 3 times the number which are nearer to the sun than the earth. Find the number of planets nearer to, and the number farther from the sun than the earth.

25. In a certain triangle, ABC, angle B is 20° more and angle C is 5° less than angle A. How large is each angle?

26. Separate the total angular magnitude around a point into three angles such that the second is 45° more than the first, and the third is 45° more than the second. (See Example 20, page 24.)

27. Certain two angles are complementary. The larger exceeds the smaller by 15. How large are the angles?

28. The complement of a certain angle is 10° more than the angle itself. What is the angle?

29. The sum of the supplement and the complement of a certain angle is 150°. What is the angle?

30. There are three angles which make up the total angle around a point. The second exceeds 3 times the first by 45°; the third is 15° less than twice the first. How large is each?

V. MULTIPLICATION

58. The law of signs for multiplication of positive and negative numbers (§ 29, page 38) may be written:

Rule. — 1. The product of two numbers having like signs is positive.

2. The product of two numbers having unlike signs is negative.

59. The law of exponents for multiplication.

Development. — 1. Review the definitions of *exponent*, *base*, and *power* of a number, given in § 22, page 26.

2. What does x^2 mean? a^3? r^7?

3. Write in exponent form:

(*a*) $b \cdot b \cdot b \cdot b$ (*b*) $m \cdot m \cdot m \cdot m \cdot m$ (*c*) $r \cdot r \cdot r \cdot r \cdot r \cdot r$

(*d*) $x \cdot x \cdot x \ . \ . \ . \ x$ (if there are 11 factors)

(*e*) $y \cdot y \cdot y \ . \ . \ . \ y$ (if there are 17 factors)

4. The product of a^3 and a^4 is found as follows:

$$a^3 = a \cdot a \cdot a; \quad a^4 = a \cdot a \cdot a \cdot a.$$

$$\therefore \ a^3 \cdot a^4 = (a \cdot a \cdot a) \times (a \cdot a \cdot a \cdot a) = a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a = a^7.$$

5. Find, as in Step 4, the following products:

(*a*) $x^2 \cdot x^4 = ?$ (*b*) $r^3 \cdot r^5 = ?$ (*c*) $m^4 \cdot m^5 = ?$

6. Observing the results in Step 5, get the following products mentally; check as in Step 4:

(*a*) $m^2 \cdot m^3 = ?$ (*c*) $x^2 \cdot x^5 = ?$

(*b*) $a^3 \cdot a^5 = ?$ (*d*) $y^6 \cdot y^3 = ?$

These examples teach the

Rule. — To find the exponent of any number in a product, add the exponents of that number in the multiplicand and multiplier.

EXERCISE 41

Using the above rule, give the following products:

1. $x^4 \cdot x^3$	**4.** $t^4 \cdot t^6$	**7.** $m^{10} \cdot m^3$	**10.** $2^3 \cdot 2^4$
2. $y^6 \cdot y^5$	**5.** $z^5 \cdot z^6$	**8.** $s^9 \cdot s^5$	**11.** $3^2 \cdot 3^3$
3. $z^3 \cdot z^7$	**6.** $k^8 \cdot k^4$	**9.** $w^8 \cdot w^7$	**12.** $10^3 \cdot 10^5$

60. Multiplication of monomials.

Development.—1. How much is $(-7)\cdot(+2)$? $(-5)\cdot(-4)$?

2. How does $9\cdot 5$ compare with $5\cdot 9$? Review §13, page 15.

3. The product of $-5x$ and $-6x^2y$ is obtained as follows:

$$(-5x)(-6x^2y) = (-5)\cdot x\cdot(-6)\cdot x^2y$$
$$= (-5)\cdot(-6)\cdot x\cdot x^2\cdot y.$$ Since the order of the factors may be changed.
$$= +30x^3y.$$

4. Find, as in Step 3, the following products:
(*a*) $3mn \times 2p$; (*b*) $(2a^3b)\times(-5ab)$; (*c*) $(-7a)\times(-2ab)$.

5. Give at sight, if possible, the following:

(*a*) $2rs\cdot 5t$
(*b*) $3xy\cdot 2z$
(*c*) $7a\cdot(-3a^2)$
(*d*) $(-4m)\cdot(-5mn)$

Rule. — To find the product of two monomials:

1. Find the product of the numerical coefficients, using the law of signs for multiplication. (§ 58.)

2. Multiply this product by the literal factors, giving to each an exponent equal to its exponent in the multiplicand plus its exponent in the multiplier.

Example. — Multiply $-5x^3yz^5$ by $+6xy^2w$.

Solution. — $(-5x^3yz^5)(+6xy^2w) = -30x^{(3+1)}y^{(1+2)}z^5w$
$$= -30x^4y^3z^5w.$$

EXERCISE 42

Find the following products:

1. $x^3 \cdot x^4$
2. $(-2x) \cdot (+3x)$
3. $(-4y)(-5y)$
4. $(+z)(-6z^2)$
5. $(-w)(-8w^2)$
6. $(-xy)(+xz)$
7. $(-a^2b)(-ab^2c)$
8. $(+mn^2)(-m^2)$
9. $(-3ab)(+2a^2)$
10. $(-4xy)(-6x^2y)$
11. $(-8ab)(+3a^2)$
12. $(-3r^2s)(-4rs^2)$
13. $(+5xy^2)(-8x^2)$
14. $(-11mn^3)(-5m^2)$
15. $(-6c^2d^3)(-4cd)$
16. $(-9m^2n^3)(+12mn^2)$
17. $(-12xy^2z)(+9xy^2)$
18. $(-6a^3b^4)(-a^3b)$
19. $(+8ax^2)(-12ax)$
20. $(+9x^2y^4)(-9xy^2)$
21. $(-7AB^2)(+13A^2B)$
22. $(-14m^2n)(-n)$
23. $(-3a)(+\frac{2}{3}a^2)$
24. $(4n)(-\frac{3}{2}n^2)$
25. $(-12)(+\frac{5}{3}r)$
26. $(-15xy)(-\frac{6}{5}x^2)$
27. $(-40)(+\frac{5}{8}t)$
28. $(-\frac{1}{2}ab) \cdot (\frac{1}{3}a^2)$
29. $(-\frac{2}{3}mn)(-\frac{3}{4}m^2)$
30. $(-\frac{3}{8}x^2)(+\frac{4}{5}xy)$

31. *a.* What does the square of any number mean?

b. Find: $(x^3)^2$; $(-5x)^2$; $(+2xy)^2$; $(-3x^2)^2$.

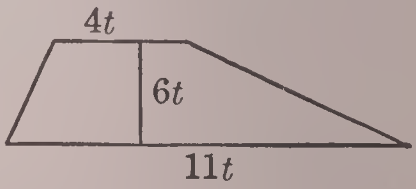
32. *a.* What is meant by the cube of a number?

b. Find: $(x^2)^3$; $(-2x)^3$; $(-3a^2)^3$; $(-\frac{1}{2}m)^3$.

33. *a.* If the base of a certain rectangle is $5x$ in. and its altitude is $3x$ in., what are its area and perimeter?

b. How much are the length, altitude, area, and perimeter of this rectangle when x is 2?

34. The base of a certain triangle is $8y$ in., and its altitude is $15y$ in. What is its area?



35. *a.* The upper base of a certain trapezoid is $4t$ in.; its lower base is $11t$ in.; its altitude is $6t$ in. What is its area? (Review § 23, page 28.)

b. What is the value of this area when t is 10?

36. The length of a certain rectangular parallelopiped is $10\,x$ in.; its width is $6\,x$ in.; its height is $4\,x$ in.

a. What is the area of its base?

b. What is the area of each of its side faces?

c. What is its volume?

37. The edge of a certain cube is $4\,x$ in. long. What is the volume of the cube?

38. What is the cost of m gallons of gasoline at p¢ each?

39. What is the cost of $2\,n$ tons of coal at c dollars each?

40. The length of a certain rectangular parallelopiped is $5\,x$ in.; its width is $3\,y$ in.; its height is $4\,z$ in.

a. What is the area of its base?

b. What is the area of one end?

c. What is the area of its front face?

Find:

41. $(.5\,m)\cdot(-3\,m^2)$

42. $(-.8\,x^2)\cdot(+.9\,xy)$

43. $(-2.5\,a^2)\cdot(-4\,ab)$

44. $(-6.8\,t)\cdot(+.3\,t^2)$

45. $(\frac{3}{4}\,r^2s)\cdot(-\frac{1}{6}\,s^2)$

46. $(-\frac{5}{8}\,mn^3)\cdot(+\frac{2}{3}\,m)$

MULTIPLICATION OF POLYNOMIALS BY MONOMIALS

61. The product of two numbers may be represented geometrically.

Example 1. — The product 5×4 may be represented by the rectangle in Fig. 1. Notice that the area is 20 square units.

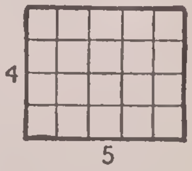


FIG. 1

Example 2. — The product $4(3 + 7)$ may be represented by Fig. 2.

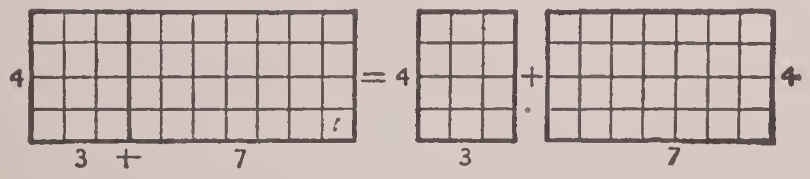


FIG. 2

$$4(3 + 7) = 4 \times 3 + 4 \times 7.$$

Check. — $4 \times 10 = 12 + 28.$

Example 3. — The product $a(b + c)$ may be represented by Fig. 3.

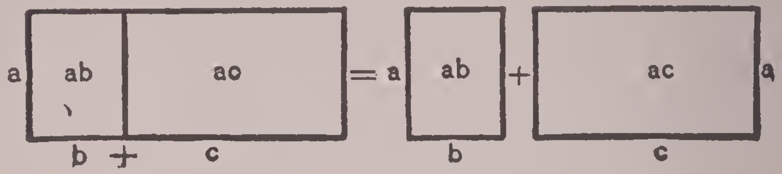


FIG. 3

$$a(b + c) = ab + ac.$$

62. Multiplying polynomials by monomials.

From the geometrical illustrations in § 61, it is clear that:

a. $5(7 + 3) = (5 \times 7) + (5 \times 3) = 35 + 15 = 50$

b. $6(4 + 5) = (6 \times 4) + (6 \times 5) = 24 + 30 = 54$

c. $a(b + c + d) = ab + ac + ad$

Rule. — To multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial and combine the results.

Example. — Multiply $3a^2 - 2ab + b^2$ by $-3ab$.

Solution. —

$$(-3ab) \times (3a^2 - 2ab + b^2) = -9a^3b + 6a^2b^2 - 3ab^3.$$

Check. — If $a = 1$ and $b = 2$, $-3ab = -6$;

$3a^2 - 2ab + b^2 = 3 - 4 + 4 = 3$; and $(-6) \times (3) = -18$.

Also, the result, $-9a^3b + 6a^2b^2 - 3ab^3, = -18 + 24 - 24 = -18$.

NOTE. — In this solution, $(-3ab)(3a^2) = -9a^3b$; $(-3ab)(-2ab) = +6a^2b^2$; and $(-3ab)(b^2) = -3ab^3$. All this should be done mentally.

EXERCISE 43

Multiply:

1. $5a - 8$ by $3a$
2. $x^2 - 4x + 5$ by $2x$
3. $7xy(x - 4y)$
4. $6(\frac{1}{2}x - \frac{1}{3}y)$

See how rapidly you can obtain correct results for the following sixteen products. Write only the results.

5. $-4x(3x^2 + 2x - 10)$
6. $5a^2(2a^3 - 5a^2 + 7)$
7. $-3s(r^2 - 2rs + s^2)$
8. $-3x^2(6x^4 - 7x^3 + 5x^2)$
9. $-2c^2(-2c^2 + 3cd + 4d^2)$

10. $-x^3(x^5 - x^4 + 5x^2)$

11. $-6ab(10a^3 - 5b^2)$

12. $x^2y(x^3 - 3x^2y + xy^2)$

13. $-rs(r^2 + rs + s^2)$

14. $-5ab(a^2 - 3ab - 7b^2)$

15. $-2mn(4m^2 - 3mn + 2n^2)$

16. $6a^3(3a^2 - 4ab + b^2)$

17. $\begin{array}{l} 5m^2 - 4mn + 3n^2 \\ \underline{2m \qquad\qquad\qquad\quad} \end{array}$

18. $\begin{array}{l} 7a^2 - 5ab + 6b^2 \\ \underline{-3ab \qquad\qquad\quad} \end{array}$

19. $\begin{array}{l} 2x^2 - 3xy + 4y^2 \\ \underline{-4xy \qquad\qquad\quad} \end{array}$

20. $\begin{array}{l} x^3 - x^2 + 3x - 4 \\ \underline{-5x \qquad\qquad\quad} \end{array}$

21. Multiply $\frac{3}{5}t + \frac{1}{2}t - \frac{2}{3}t$ by 60, collect terms, and find the value of the result when $t = 2$.

Solution. — 1. $60\left(\frac{3}{5}t + \frac{1}{2}t - \frac{2}{3}t\right) = \overset{12}{\cancel{60}} \cdot \frac{3}{\cancel{5}}t + \overset{30}{\cancel{60}} \cdot \frac{1}{\cancel{2}}t - \overset{20}{\cancel{60}} \cdot \frac{2}{\cancel{3}}t$

2. $= 36t + 30t - 40t$

3. $= 26t.$

4. When $t = 2$, $26t = 26 \cdot 2$, or 52.

Multiply as in Example 21, and find the value of:

22. $12\left(\frac{x}{2} - \frac{x}{3} + \frac{x}{4}\right)$ when $x = 5$.

23. $20\left(\frac{y}{2} - \frac{y}{5} + \frac{y}{4}\right)$ when $y = 6$.

24. $15\left(\frac{z}{5} - \frac{z}{3} + \frac{z}{15}\right)$ when $z = -3$.

25. $30(\frac{3}{10}t - \frac{1}{15}t + \frac{1}{6}t)$ when $t = 10$.

26. $40(\frac{5}{8}x - \frac{3}{20}x - \frac{7}{10}x)$ when $x = -4$.

27. $36(\frac{1}{18}w - \frac{2}{3}w - \frac{1}{2})$ when $w = 8$.

28. $28(\frac{3}{14}a + \frac{5}{7}b - \frac{1}{4}c)$ when $a = 2$, $b = 3$, and $c = 9$.

29. $24(\frac{5}{8}x - \frac{3}{4}y + \frac{1}{6}z)$ when $x = 2$, $y = 1$, and $z = 5$.

30. *a.* The base of a certain rectangle is $(x + 8)$ in., and its altitude is $3x$ in. What is its area?

b. What is this area when $x = 2$?

31. *a.* The altitude of a certain triangle is $(y - 3)$ in., and its base is $12y$ in. What is its area?

b. What is this area when $y = 5$?

32. *a.* There are n articles which cost $(n + 15)$ cents each. What is their total value?

b. What is this value when $n = 10$?

33. *a.* A train travels t hours at the rate of $(10t + 5)$ miles per hour. How far does it travel?

b. What is this distance when $t = 4$?

34. Simplify $5y - 3(x - 4)$.

Solution. — 1. 3 times $(x - 4)$ is to be subtracted from $5y$.

2. But, by § 28, this is the same as adding $-3(x - 4)$ to $5y$.

3. $\therefore$ simply multiply $(x - 4)$ by -3 and write the results. Then

$$\begin{aligned} &5y - 3(x - 4) \\ &= 5y - 3x + 12. \end{aligned}$$

Simplify the following expressions, combining like terms.

35. $5y - 2(3y - 4)$

36. $6s + 8(4s - 2)$

37. $7t - 5(6 - 3t)$

38. $-8w + 9(2w - 4)$

39. $-13x - 9(5 - 3x)$

40. $(3z - 7) - 6(4z - 10)$

41. $9(11 - 7k) - 8(12 - 5k)$

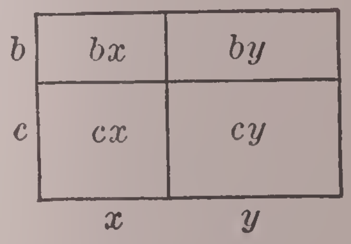
42. $12(\frac{2}{3}m + 4) + 9(\frac{1}{3}m - 5)$

43. $10(\frac{7}{2} - 3s) - 8(4 - 5s)$

44. $16(\frac{3}{4}c + \frac{5}{8}) - 6(2c - 3)$

63. Multiplying a polynomial by a polynomial.

The product $(x + y)(b + c)$ may be represented by the figure at the right. $(x + y)$ is the base, and $(b + c)$ is the altitude. The areas of the four parts are marked.



$$\therefore \ (x + y)(b + c) = bx + by + cx + cy$$

Rule. — **To multiply one polynomial by another:**

1. Arrange multiplier and multiplicand in ascending or descending powers of the same letter.

2. Multiply the multiplicand by each term of the multiplier.

3. Add the partial products.

Example. — Multiply $3a - 4b$ by $2a - 5b$, and check the solution by letting $a = 1$ and $b = 2$.

$$\begin{array}{lll}
\textit{Solution.}\ — & 3a - 4b & \\
 & \underline{2a - 5b} & \\
 & 6a^2 - 8ab & \text{This is } 2a \times (3a - 4b) \\
 & \underline{\qquad - 15ab + 20b^2} & \text{This is } -5b \times (3a - 4b) \\
\text{Adding,} & 6a^2 - 23ab + 20b^2 & \text{This is the product.}
\end{array}$$

Check. — a and b may be any numbers.

$$\text{When } a = 1 \text{ and } b = 2, \left.\begin{array}{r} 3a - 4b = 3 - 8 = -5. \\ 2a - 5b = 2 - 10 = -8. \end{array}\right\} \begin{array}{l} \text{And } (-5)(-8) \\ \quad = +40. \end{array}$$

The product, $6a^2 - 23ab + 20b^2 = 6 - 46 + 80$; or 40.

Since the two results are equal, the product is probably correct.

EXERCISE 44

Multiply:

1. $x + 9$ by $x + 7$
2. $x + 11$ by $x + 13$
3. $y - 7$ by $y - 5$
4. $z - 8$ by $z - 10$
5. $w + 4$ by $w - 12$
6. $w - 16$ by $w + 3$
7. $2z + 5$ by $3z + 2$
8. $6m - 3$ by $4m + 5$
9. $4t - 5$ by $6t - 2$
10. $8x - 3$ by $5x - 4$
11. $3a + 2b$ by $2a - b$
12. $5m - 6t$ by $3m + 4t$
13. $7w + 6p$ by $8w - 4p$
14. $9x^2 - y^2$ by $2x^2 + 3y^2$
15. $3w^2 - 7m$ by $4w^2 + 5m$
16. $\frac{1}{2}a + \frac{1}{3}b$ by $2a + b$
17. $\frac{1}{2}x - \frac{1}{3}y$ by $\frac{1}{2}x + \frac{1}{3}y$
18. $\frac{2}{3}x - 3z$ by $\frac{2}{3}x - 3z$
19. $\frac{3}{2}m - \frac{2}{3}t$ by $\frac{3}{2}m - \frac{2}{3}t$
20. $\frac{1}{3}x + \frac{1}{4}y$ by $\frac{1}{3}x - \frac{1}{4}y$
21. $(x^2 + 2x - 3)(x - 1)$
22. $(x^2 + xy + y^2)(x - y)$
23. $(x^2 - xy + y^2)(x + y)$
24. $(m^2 - 4m + 3)(m - 2)$
25. $(a^2 - 2a - 4)(2a + 3)$
26. $(x^2 - 3xy + 2y^2)(x - 2y)$
27. $(3r^2 + 6r - 8)(4r + 3)$

28. $(4m^2 - 2mn + n^2)(m - 3n)$

29. $(4s^2 + r^2 - 4rs)(r - 2s)$

30. $(3c^2 + d^2 - 2cd)(2c + d)$

31. $(4n^2 - 7 + 5n)(n^2 - 2n + 1)$

32. $(5t + 2t^2 - 6)(3t^2 - 5t + 4)$

33. $(a^2 - 3 - 2a)(a^2 + 3 - 2a)$

34. $(5x - 7 + 4x^2)(3 + 2x^2 - 3x)$

35. $(9r^2 + 4s^2 + 6rs)(9r^2 + 4s^2 - 6rs)$

36. $(x^2 - 2xy + y^2)(x^2 + 2xy + y^2)$

37. $(x^3 - x^2y + xy^2 - y^3)(x + y)$

38. $(a^3 + 2a^2b + 4ab^2 + 8b^3)(a - 2b)$

39. $(r + s - t)(r - s + t)$

40. $(2a - 3b + c)(2a - 3b - c)$

41. $(3x - 2y)^2$

42. $(x - 3y)^3$

43. $(4r - 5s)^2$

44. $(2m - 3n)^3$

45. $(x + y)^4$

46. $(x - y)^4$

47. *a.* What is the area of the rectangle whose base is $(5x + y)$ in., and whose altitude is $(2x - y)$ in.?

b. How much is this area when $x = 3$ and $y = 4$?

48. *a.* What is the volume of the cube whose edge is $(2a - b)$ in.?

b. What is this volume when $a = 4$ and $b = 3$?

49. *a.* What is the cost of $(2n + m)$ articles at $(n - m)$¢ each?

b. How much is this result when $n = 8$ and $m = 3$?

50. *a.* A man traveled $(n + 8)$ hr. at the rate of $(m + 2)$ mi. per hour. How far did he go?

b. What were his rate, time, and distance when $m = 30$ and $n = 3$?

NOTE. — Supplementary examples in multiplication are on page 431.

64. Multiplication used in solving equations and problems.

Example. — Seven times the complement of a certain angle exceeds twice its supplement by 20°. Find the angle. (Review Examples 4 and 15, page 23.)

Solution. — 1. Let $a =$ the number of degrees in the angle.
2. Then $90 - a =$ the number of degrees in its complement,
3. and $180 - a =$ the number of degrees in its supplement.
4. $\therefore\ 7(90 - a) - 2(180 - a) = 20.$
5. Multiplying, $630 - 7a - 360 + 2a = 20.$
6. C. T. $-5a + 270 = 20.$
7. S_{270} $-5a = -250.$
8. M_{-1} $5a = 250.$
9. D_5 $a = 50$, the number of degrees in the angle.

Check. — The complement of the angle is 40°, and the supplement is 130°. Does $7 \times 40 - 2 \times 130 = 20$? Does $280 - 260 = 20$? Yes.

NOTE. — In Step 8, both members are multiplied by -1 to change $-5a$ into $+5a$.

In Step 5, 180 multiplied by -2 gives -360; $-a$ multiplied by -2 gives $+2a$. (Review Example 34, page 80.)

You will need to take similar steps in the following examples.

EXERCISE 45

Solve and check the following equations:

1. $3(x - 4) = 33$
2. $2(3y - 8) = 20$
3. $5(2x - 1) = 35$
4. $6(8x - 1) = 4x + 16$
5. $7(1 - 9t) = 12t - 18$
6. $3(1 - 12m) = 4m - 7$
7. $4(y - 3) - 5 = 3$
8. $6(z + 2) - 40 = 50$
9. $8(3t - \frac{1}{2}) + 9 = 17$
10. $2(3r - 11) = 5(11 - r)$
11. $4(16 - m) = 3(m - 9)$
12. $2(7x + 1) = 3(7x - 4)$
13. $3(2x - 4) - 2(x - 5) = 2(x + 6)$
14. $6(2 - 3y) + 30 = 3(4y - 6)$
15. $14 - 5(3a - 2) = 3(a - 10)$

16. $10 - 5(r - 4) = 6(\frac{13}{2} - r)$
17. $2(9 - 3v) + 3(4 - v) = 21$
18. $7(g - 8) - 6(2g - 15) = 4(6 - g)$
19. $3(5 - n) - 4(2n - 7) + 3(7 + n) = 0$
20. $4(3m - 5) + 3(6m - 1) = 3(9m - 7) - 1$

21. The greater of certain two numbers is 7 more than the smaller. Twice the greater number is 6 more than three times the smaller. Find the numbers.

22. One number exceeds a certain other number by 5. Six times the smaller number, diminished by four times the larger number, is 6. What are the numbers?

23. Separate the number 35 into two parts such that twice the larger, diminished by three times the smaller, is 10.

24. The sum of certain two numbers is 45. If four times the smaller be decreased by twice the larger, the result is 12. What are the numbers?

25. One number exceeds another number by 3. Six times the larger, diminished by seven times the smaller, is 7. What are the numbers?

26. There are two consecutive integers such that the sum of twice the larger and four times the smaller is 110. What are these integers?

27. There are certain two consecutive integers such that the square of the larger minus the square of the smaller gives 25. Find these integers.

28. There are three consecutive integers such that three times the first, plus two times the second, plus the third, gives 100. Find these integers.

29. There are three consecutive even integers such that the first, plus twice the second, plus three times the third, gives 100. What are they?

30. The base of a certain rectangle is 8 in. more than the side of a certain square; its altitude is 3 in. less than the side of the square. The perimeter of the rectangle is 74 in. What is the length of a side of the square, and what are the base and altitude of the rectangle?

31. If 5 times the complement of a certain angle be subtracted from 2 times its supplement, the result is 15°. What is the angle?

32. If the supplement of a certain angle be subtracted from four times its complement, the remainder is 5° more than four times the angle itself. What is the angle?

EXERCISE 46

1. A is now 18 years old. Express his age:

a. 6 years ago; *b.* n years ago; *c.* x years from now.

2. B is x years old. Express his age:

a. 4 years hence (from now); *b.* 6 years ago; *c.* y years ago.

3. A is now x years old. B is 8 years older. Express:

a. B's present age; *b.* the sum of their ages;

c. the age of each 10 years ago;

d. the age of each 5 years hence.

4. A is now n years old. B is four times as old. Express:

a. B's present age; *b.* the age of each 4 years hence;

c. the fact that B's age as found in part *b* is 3 times A's age.

d. Solve the equation formed in part *c*. Obtain the age of each.

5. B is three times as old as A is now. Make a table and supply the answers to the questions asked below.

a. Let a = the number of years in A's age now.

	Now	10 Years Ago
A's age	?	?
B's age	?	?

b. Write the equation which expresses the fact that B's age ten years ago was 5 times A's age then. Solve it, and obtain the ages of A and B.

c. Check the solution by finding their ages ten years ago, and comparing them with the statement in the problem.

Solve the following problems:

6. A is now four times as old as B. In six years he will be twice as old as B is then. How old is each now?

7. C is now five times as old as D. In fifteen years he will be twice as old as D is then. How old is each?

8. A is now 12 years older than B. Four years from now, A will be twice as old as B is then. What are their present ages?

9. The sum of the ages of a father and son now is 30 years. In 9 years the father's age will be only 3 times the son's age then. How old is each now?

10. A's age is now 38 years and B's is 5 years.

a. How old will each of them be in x years?

b. If A will be 4 times as old as B at that time, form the equation from the ages obtained in part *a*, and solve it for x.

11. A's age is now 33 years, and B's is 3 years. In how many years will A's age be just 6 times B's age at the same time?

12. A's age now is 65 years, and B's is 20 years. How many years ago was A 6 times as old as B was?

Equations for problems may often be written at sight.

Thus, for Example 6 we can write at once:

$$(4b + 6) = 2(b + 6).$$

Write at sight the equations for the following problems, and solve them as rapidly as you can. Time yourself.

13. A is four times as old as B now. Five years ago, he was 7 times as old as B was then. Find their ages.

14. A is 5 times as old as B. Five years ago, he was 4 times as old as B will be one year hence. Find their ages.

15. A is now 6 years older than B. Three years ago, A was twice as old as B was then. Find their ages.

16. A is now 10 years older than B. Five years from now he will be 5 times as old as B was five years ago. Find their ages.

17. A is now 7 times as old as B. A's age in four years will be 5 times what B's age will be in two years. Find their ages.

EXERCISE 47

Mixture and Coin Problems

1. A grocer has some 90¢ tea and some 40¢ tea. How many pounds of each must he take to form a mixture of 100 pounds, which he can sell for 60¢ per pound?

Solution. — 1. Let $n =$ the number of pounds he must take, of the 90¢ tea.
2. $\therefore\ 90\,n$¢ $=$ the value of these n pounds.
3. Then $100 - n =$ the number of pounds he must take, of the 40¢ tea.
4. $\therefore\ 40(100 - n)$¢ $=$ the value of these $(100 - n)$ pounds.
5. $\therefore\ 90\,n + 40(100 - n) =$ the total value of the tea.
6. Also 100×60¢ $=$ the total value of the tea.
7. $\therefore\ 90\,n + 40(100 - n) = 100 \times 60.$

Complete the solution, obtaining n. Answer the questions asked in the problem, and check your solution.

NOTE. — The essential statements in Steps 2–6 can be arranged thus:

KIND OF TEA	NO. OF POUNDS	VALUE IN CENTS
90¢ tea	n	$90\,n$¢
40¢ tea	$100 - n$	$40(100 - n)$¢
60¢ tea	100	100×60¢

Many teachers advise pupils to substitute this table for Steps 2–6 of the solution above.

2. A grocer wishes to make a mixture of 60 pounds of coffee to sell at 40¢ per pound. He has some 55¢ coffee and some 30¢ coffee. How many pounds of each should he take?

3. A seedsman wishes to make a mixture consisting of clover seed and timothy seed which he can sell at 11¢ per pound. His clover seed is worth 23¢ per pound and his timothy 9¢ per pound. How many pounds of each should he take for a batch of 200 pounds?

4. A grocer has 20 pounds of 60¢ coffee. How many pounds of 30¢ coffee should he mix with it so that the mixture can be sold at 35¢ per pound?

5. A dealer has walnuts valued at 43¢ per pound and almonds valued at 35¢ per pound. How many pounds of each should he take to make a mixture of 150 pounds which he can sell at 40¢ per pound?

6. A sum of money amounting to $2.80 consists of pennies, nickels, and quarters. The number of nickels is 3 times the number of pennies, and the number of quarters is 3 more than the number of pennies. How many of each coin are there?

Solution. — 1. Let $p =$ the number of pennies,
2. then $p¢ =$ their value in cents.
3. $\therefore\ 3p =$ the number of nickels,
4. and $15p¢ =$ their value in cents.
5. $\therefore\ p + 3 =$ the number of quarters,
6. and $25(p + 3)¢ =$ their value in cents.
7. $\therefore\ p + 15p + 25(p + 3) =$ the total value of the coins in cents.
8. Also $280 =$ the total value of the coins in cents.
9. $\therefore\ p + 15p + 25(p + 3) = 280.$

Complete the solution, obtaining p, and check your solution.

NOTE. — The essential statements in Steps 2–8 can be arranged thus:

KIND OF COIN	NUMBER OF THEM	VALUE OF THEM IN CENTS
Pennies	p	p
Nickels	$3p$	$15p$
Quarters	$p + 3$	$25(p + 3)$
All together		280

Many teachers advise pupils to substitute this table for Steps 2–8 of the solution above.

7. A sum of money amounting to \$2.55 consists of nickels and dimes. The number of dimes is 6 more than the number of nickels. How many of each kind of coin are there?

8. A sum of money amounting to \$5.35 consists of nickels, dimes, and quarters. There are 5 more quarters than there are nickels; the number of dimes is 1 less than three times the number of nickels. How many of each kind of coin are there?

9. A sum of money amounting to \$38.75 consists of quarters, half-dollars, \$1.00 bills, and \$2.00 bills. The number of half-dollars is 2 more than the number of quarters; the number of \$1.00 bills is 3 less than twice the number of quarters; the number of \$2.00 bills is 1 less than the number of quarters. How many of each kind of coin are there?

10. A girl's savings bank had in it pennies, nickels, dimes, and quarters. Their total value was \$10.16. The number of dimes was twice the number of quarters. There were 15 more nickels than there were dimes. The number of pennies was 3 less than four times the number of quarters. How many of each kind of coin were there?

65. Equations having fractional coefficients.

Example. — If the sum of twice a certain number and one half of the number be diminished by two thirds of the number, the result is $27\frac{1}{2}$. What is the number?

Solution. — 1. Let n = the number.

2. Then $2n + \frac{1}{2}n - \frac{2}{3}n = 27\frac{1}{2}$.

3. To obtain an equation which does not have in it any fractions, multiply by the smallest number which contains all the denominators as divisors. This is called the *least common multiple of the denominators.*

M_6 $\quad 6(2n + \frac{1}{2}n - \frac{2}{3}n) = 6 \times \frac{55}{2}$.

4. $\quad \therefore 12n + 3n - 4n = 165$.

5. C. T. $\quad 11n = 165$.

6. D_{11} $\quad n = 15$, the number.

$$\overset{3}{\cancel{6}} \times \frac{55}{\cancel{2}} = 165$$

$$\overset{2}{\cancel{6}} \times \frac{2}{\cancel{3}}n = 4n$$

Check. — Twice the number = 30; one half the number = $7\frac{1}{2}$; two thirds the number = 10. Does $30 + 7\frac{1}{2} - 10 = 27\frac{1}{2}$? Yes.

EXERCISE 48

Solve the following equations:

1. $\frac{x}{2} - \frac{x}{5} = \frac{9}{10}$
2. $y + \frac{y}{3} = \frac{20}{3}$
3. $\frac{2x}{5} - \frac{x}{4} = \frac{9}{10}$
4. $\frac{5t}{2} - \frac{1}{3} = \frac{2t}{3}$
5. $\frac{2r}{3} = \frac{r}{4} + \frac{5}{3}$
6. $\frac{x}{4} = \frac{9}{10} + \frac{x}{5}$
7. $\frac{3s}{8} - \frac{s}{3} = \frac{2}{3}$
8. $\frac{4w}{5} = \frac{2w}{3} + 2$
9. $\frac{5m}{6} - \frac{26}{5} = \frac{2m}{5}$
10. $\frac{x}{5} = \frac{x}{4} - \frac{1}{2}$
11. $\frac{r}{2} + \frac{3r}{4} - \frac{2r}{3} = 7$
12. $t - \frac{3t}{4} = 10 - \frac{t}{6}$
13. $\frac{c}{3} - \frac{13}{6} = \frac{c}{2} - \frac{3c}{5}$
14. $3d - \frac{5d}{6} = \frac{d}{2} + \frac{5}{9}$
15. $\frac{7m}{8} - \frac{m}{5} = \frac{3}{40} - \frac{3m}{10}$
16. $c + \frac{c}{7} = \frac{62}{7} - \frac{c}{3}$
17. $a = 12 - \frac{a}{3} - \frac{a}{6}$
18. $\frac{3s}{2} - \frac{7}{3} = \frac{4s}{5} - \frac{7}{5}$
19. $2x - \frac{5x}{3} = \frac{3x}{2} - \frac{7}{16}$
20. $\frac{3w}{5} - \frac{4w}{3} = \frac{4}{9} - w$

EXERCISE 49

1. Two thirds of a certain number is 20 more than one fourth of the number. What is the number?

2. What number is such that eight fifths of it, decreased by 50, gives a remainder of fourteen fifteenths of it?

3. What number increased by one eighth of itself equals the sum of 30 and three sixteenths of itself?

4. One half a certain number equals the excess of five sevenths of the number over 6. What is the number?

5. What number exceeds by 13 the sum of one third itself, one fourth itself, and one fifth itself?

6. There are three consecutive integers such that the sum of the second and third exceeds one fourth of the first by 17. What are these integers?

7. There are three consecutive integers such that the sum of the second and third is 21 more than one fifth of the first. What are these integers?

8. The perimeter of a certain triangle is 41 in. The second side is nine sevenths of the first side; the third side is one half the second side. How long is each side?

9. The perimeter of a certain quadrilateral is 47 in. One side is one fourth of the longest side; a second side is two fifths of the longest; the third is seven tenths of the longest. How long is each side?

10. There are four angles whose sum is the total angle around a point. (See Example 20, page 24.) The second angle is five sevenths of the first; the third is one half of the first; and the fourth is one half of the second. How large is each angle?

11. Certain two angles are complementary. (See Example 4, page 23.) The smaller is two sevenths of the larger. How large are these angles?

12. Certain two angles are supplementary. (See Example 15, page 24.) The smaller is seven elevenths of the larger. How large is each of these angles?

13. In certain triangles having three unequal angles, the smallest angle is one sixth of the largest, and the remaining angle is one third of the largest. How large is each of the angles? (See Example 22, page 25.)

14. In other triangles, the smallest angle is three tenths of the largest, and the remaining angle is one half of the largest. How large is each of the angles?

15. The perimeter of the Treasury at Washington, D. C., is 1400 ft. Its width is five ninths of its length. How long and how wide is it?

16. A's age is now two thirds B's age. A's age 4 years from now will be five sixths of B's present age. How old is each now?

17. A's present age is one third of B's. A's age 10 years from now will be what B's age was 10 years ago. How old is each now?

18. A man has $4.50 in nickels, dimes, and quarters. He has three fourths as many dimes as he has nickels, and five eighths as many quarters as he has nickels. How many coins of each kind has he?

19. The total value of a sum of money consisting of nickels, quarters, and half-dollars is $16.35. The number of half-dollars is two thirds of the number of nickels. The number of quarters is four thirds of the number of half-dollars. How many of each coin has he?

20. A grocer has a mixture of two kinds of coffee, having a total value of $49.50. One kind was worth 30¢ per pound, and the other, 45¢ per pound. The number of pounds of the latter used in the mixture was one fourth of the number of pounds of the former. How many pounds of each did he use?

VI. DIVISION

66. Division is the process of finding one of two numbers when their product and the other number are given.

To divide 15 by 3 means to find the number by which 3 must be multiplied to give the product 15.

The **dividend** is the product of the numbers; it is the number divided.

The **divisor** is the other given number; it is the number by which the dividend is divided.

The **quotient** is the required number.

67. It is clear that $a \div a = 1$; for $a \times 1 = a$.

68. Division by zero is impossible.

Thus, if we try to find the quotient of $6 \div 0$, and let q equal the quotient, we should have the relation

$$6 = 0 \cdot q.$$

But $0 \cdot q = 0$ and not 6, so there cannot be any number to use as q. Hence, there is no number to represent the quotient of $6 \div 0$.

69. Division is **indicated** by writing a fraction whose numerator is the dividend and whose denominator is the divisor.

Thus, the quotient of $15 \div 5$ is written $\frac{15}{5}$.

The quotient of $7\,abc \div 3\,xy$ is written $\dfrac{7\,abc}{3\,xy}$.

Note. — The line — was used to indicate division long before the symbol $\div$.

70. Division of a product by a number.

Example. — Divide 6×8 by 2.

Solution. —

1. If 6 is divided by 2, $\dfrac{6 \times 8}{2} = \dfrac{\overset{3}{\cancel{6}} \times 8}{\cancel{2}} = 24.$

2. If 8 is divided by 2, $\frac{6 \times 8}{2} = \frac{6 \times \overset{4}{\cancel{8}}}{\cancel{2}} = 24.$

3. If both 6 and 8 are divided by 2,

$$\frac{6 \times 8}{2} = \frac{\overset{3}{\cancel{6}} \times \overset{4}{\cancel{8}}}{\cancel{2}} = 12.$$

4. Since we know that $\frac{6 \times 8}{2} = 48 \div 2$ or 24, it is clear that the results obtained in Steps 1 and 2 are correct, but that the result in Step 3 is incorrect. Hence,

Rule. — To divide the product of two or more numbers by a number, divide any one of the factors by the number, but divide only one of them by it.

EXERCISE 50

Find each of the following indicated divisions in two ways:

1. $\frac{9 \times 12}{3}$
2. $\frac{18 \times 24}{6}$
3. $\frac{28 \times 56}{7}$

71. The law of signs for division.

Since $(+2) \times (+3) = +6$, then $(+6) \div (+2) = +3$.

Since $(-2) \times (+3) = -6$, then $(-6) \div (-2) = +3$.

Since $(+2) \times (-3) = -6$, then $(-6) \div (+2) = -3$.

Since $(-2) \times (-3) = +6$, then $(+6) \div (-2) = -3$.

If the signs of the dividend, the divisor, and the quotient in each of the previous statements are examined, the following rules become clear:

Rule. — 1. The quotient of two numbers having like signs is positive.

2. The quotient of two numbers having unlike signs is negative.

EXERCISE 51

Find:

1. $(+18) \div (+3)$
2. $(+12) \div (+2)$
3. $(+15) \div (+5)$
4. $(-30) \div (-6)$
5. $(-45) \div (-9)$
6. $(-50) \div (-5)$
7. $(+18) \div (-2)$
8. $(-24) \div (+3)$
9. $(-36) \div (-6)$
10. $(+42) \div (-7)$
11. $(-100) \div (-20)$
12. $(+72) \div (-12)$
13. $(-81) \div (+3)$
14. $(-90) \div (-10)$
15. $(-84) \div (+7)$
16. $(+60) \div (-10)$
17. $(+90) \div (-30)$
18. $(+6) \div (-12)$
19. $(-10) \div (-20)$
20. $(-5) \div (+4)$
21. $(-15) \div (-15)$
22. $(+32) \div (-32)$
23. $(-18) \div (-1)$
24. $(+17) \div (-1)$
25. $(-108) \div (-3)$
26. $(+120) \div (-8)$
27. $(-33) \div (-11)$
28. $(-64) \div (+16)$
29. $(+80) \div (-5)$
30. $(-39) \div (-13)$
31. $(+16) \div (-16)$
32. $(-91) \div (-7)$
33. $(-200) \div (+5)$
34. $(+6) \div (-18)$
35. $(-7) \div (-35)$
36. $(-4) \div (+6)$
37. $(-5) \div (-8)$
38. $(-9.6) \div (-3)$
39. $(+5.8) \div (-2)$
40. $(-6.4) \div (-2)$

72. The law of exponents for division.

Development. — 1. Review the definitions of *exponent*, *base*, and *power* of a number in § 22.

2. Divide a^5 by a^3.

Solution. — $$\frac{a^5}{a^3} = \frac{\overset{1}{\cancel{a}} \cdot \overset{1}{\cancel{a}} \cdot \overset{1}{\cancel{a}} \cdot a \cdot a}{\underset{1}{\cancel{a}} \cdot \underset{1}{\cancel{a}} \cdot \underset{1}{\cancel{a}}} = 1 \cdot a \cdot a = a^2.$$

Therefore $a^5 \div a^3 = a^2$. *Check.* — $a^3 \cdot a^2 = a^5$.

Each a in the denominator is divided into one of the a's in the numerator. The quotient in each case is 1, since $a \div a = 1$.

3. Find as in Step 2 the following quotients:

a. $y^6 \div y^4 = ?$ *b.* $m^6 \div m^4 = ?$ *c.* $t^{10} \div t^6 = ?$

4. Examine carefully the exponents in the dividend, the divisor, and the quotient. In the following problems, try to give the results immediately without going through the solution as in Step 2. Test by multiplication.

a. $p^8 \div p^6 = ?$ *c.* $b^7 \div b^4 = ?$

b. $a^6 \div a^5 = ?$ *d.* $c^8 \div c^4 = ?$

5. Divide a^4b^6 by a^3b^2.

Solution. — $a^4b^6 \div a^3b^2 = \dfrac{a^4b^6}{a^3b^2} = \dfrac{\overset{1}{\cancel{a}} \cdot \overset{1}{\cancel{a}} \cdot \overset{1}{\cancel{a}} \cdot a \cdot \overset{1}{\cancel{b}} \cdot \overset{1}{\cancel{b}} \cdot b \cdot b \cdot b \cdot b}{\underset{1}{\cancel{a}} \cdot \underset{1}{\cancel{a}} \cdot \underset{1}{\cancel{a}} \cdot \underset{1}{\cancel{b}} \cdot \underset{1}{\cancel{b}}} = ab^4.$

Rule — **The exponent of any number in the quotient is equal to its exponent in the dividend minus its exponent in the divisor.**

Historical Note. — This rule was known to Stifel.

Notice that $b^2 \div b^2 = b^{2-2} = b^0$ by the law of exponents.

Since $b^2 \div b^2$ must equal 1, we agree that $b^0 = 1$.

The zero power of any number is 1.

Thus: $a^0 = 1$; $5^0 = 1$; $c^0 = 1$.

73. Division of monomials by monomials.

Example 1. — Divide $54\,a^7b^2c^3$ by $-9\,a^4b^2c^2$.

Solution. — $\dfrac{54\,a^7b^2c^3}{-9\,a^4b^2c^2} = -6\,a^{(7-4)}b^{(2-2)}c^{(3-2)}$

$= -6\,a^3b^0c$, or $-6\,a^3c$.

Check. — $(-6\,a^3c) \cdot (-9\,a^4b^2c^2) = +54\,a^7b^2c^3$.

Rule. — To divide a monomial by a monomial:

1. Make the quotient positive, if the dividend and divisor have like signs; make it negative, if they have unlike signs.

2. Find the quotient of the absolute values of the numerical coefficients.

3. Multiply the quotient of Step 2 by the product of the literal factors, giving each its exponent in the dividend minus its exponent in the divisor.

4. Omit any literal factor which has the same exponent in the dividend and divisor.

Example 2. — Divide $-33a^6bx^2y^4$ by $+3a^5x^2y$.

Solution. — $(-33a^6bx^2y^4) \div (+3a^5x^2y) = -11aby^3$.

Check. — These solutions may be checked by substitution, for they must be correct for all values of the literal numbers (except 0 sometimes). A better way is to use the rule that the divisor times the quotient equals the dividend.

Here, does $(+3a^5x^2y) \times (-11aby^3) = -33a^6bx^2y^4$? Yes.

EXERCISE 52

Divide:

1. y^6 by y^4
2. m^7 by m^2
3. r^8 by r^5
4. t^9 by t^4
5. a^3b^2 by ab
6. r^4s^3 by r^2s
7. x^6y^3 by x^2y^3
8. m^4n^6 by m^4n
9. x^5y^3 by x^4y^3
10. m^3n^2 by mn^2
11. $12a^3$ by $3a^2$
12. $18x^4$ by $6x^2$
13. $20r^3s^2$ by $5r^2s$
14. $28c^4d^3$ by $7c^2d$
15. $-14mn$ by $-7n$
16. $+18a^2b$ by $-3ab$
17. $-12xy^3$ by $-6xy^2$
18. $-24r^3t$ by $+8r^2$
19. $-34m^2n^3$ by $+17mn^3$
20. $+40x^5y^6$ by $-8x^3y^2$
21. $-32a^3b^4c$ by $-4ab^4c$
22. $35x^5y^2z^3$ by $-7x^3y^2$
23. $-44a^5xy^3$ by $-4a^4y^2$
24. $75x^5y^4z^3$ by $-15xy^3$
25. $42a^3b^2c^2$ by $-14ab^2c^2$
26. $-64x^8y^6$ by $-8x^7y^6$
27. $-42ab^3c^5$ by $-7b^2c^4$
28. $-60x^4y^5z^6$ by $-12x^2y^3z^4$
29. $-60a^4b^5c^3$ by $-20a^4b^3$
30. $+84c^5d^6$ by $-12c^4d$
31. $-51ab^3c^2$ by $-3ac^2$
32. $+48r^5s^6$ by $-16r^4s^6$
33. $5ab$ by $10a$
34. $4mn^2$ by $8m$
35. $-3r$ by $-15r$
36. $+9cd$ by $-12c$
37. $2a^2 \div \frac{1}{3}a$
38. $(-3cd) \div (\frac{1}{2}c)$
39. $3^5 \div 3^4$
40. $2^6 \div 2^3$

74. Division of polynomials by monomials.

Development. — 1. Since $2 \times 9 = 18$, then $\frac{18}{2} = 9$.

2. Since $2(x + 3) = 2x + 6$, then $\dfrac{2x + 6}{2} = x + 3$.

3. Since $3(a - 5) = 3a - 15$, what does $\dfrac{3a - 15}{3}$ equal?

4. What does each of the following equal? Test the result by multiplying the divisor and quotient.

a. $\dfrac{2x + 2}{2}$; *b.* $\dfrac{6r + 4}{2}$; *c.* $\dfrac{6m - 9}{3}$.

5. Since $a(b + c) = ab + ac$,

Then $\dfrac{ab + ac}{a} = b + c$.

Rule. — To divide a polynomial by a monomial:

1. Divide each term of the dividend by the divisor.

2. Unite the results with their signs.

Example 1. — Divide $12x^3 - 6x^2 + 3x$ by $-3x$.

Solution. — $\dfrac{12x^3 - 6x^2 + 3x}{-3x} = -4x^2 + 2x - 1$.

Check. — $(-4x^2 + 2x - 1) \cdot (-3x) = 12x^3 - 6x^2 + 3x$.

Example 2. — Divide $-9a^3 + 3a^2 - 12a^4$ by $-3a^2$.

Solution. — $(-12a^4 - 9a^3 + 3a^2) \div (-3a^2) = 4a^2 + 3a - 1$.

Check. — Multiply the quotient by the divisor; the result should equal the dividend.

EXERCISE 53

Divide:

1. $5r - 10s$ by 5

2. $12p - 8q$ by 4

3. $15x^2 - 18y^2$ by 3

4. $9x^2 - 7x$ by x

5. $18m^3 - 12m$ by $3m$

6. $6x^4 - 3x^3$ by $-3x$

7. $-20r^3 + 14r^2s$ by $-2r^2$

8. $12m^2n - 18mn^2$ by $-6mn$

9. $-33a^2b + 44ab^2$ by $-11ab$

10. $28c^3d - 21cd^3$ by $-7cd$

11. $-72r^6 - 80r^4 + 64r^2 - 32r$ by $-8r$

12. $15x^3 - 18x^2y + 21xy^2$ by $-3x$

13. $-48x^{10} + 96x^8 - 72x^6$ by $-12x^5$

14. $12y^5 - 8y^4 + 16y^3$ by $-4y^2$

15. $60m^9 - 45m^6 + 30m^3$ by $-15m^3$

16. $33a^3b^3c^3 - 44a^2b^2c^2 + 55abc$ by $-11abc$

17. $25m^7n^6 - 30m^3n^4 + 40m^2n^3$ by $-5mn^2$

18. $15c^4d^2 - 20c^3d^3 - 25c^2d^4$ by $-5c^2d^2$

19. $6a^2b^2c^3 - 9a^3b^3c^4 + 12a^4bc^2$ by $-3a^2bc^2$

20. $-49x^3y^4z^2 + 56x^2y^3z^4$ by $-7x^2y^2z^2$

21. $\dfrac{a^5b - a^3b^3 + ab^5}{ab}$

22. $\dfrac{x^4 - 12x^3 + 6x^2}{x^2}$

23. $\dfrac{-m^2 + m - 1}{-1}$

24. $\dfrac{-2m^2 + 4mn - 6n^2}{-2}$

25. $\dfrac{16x^3y - 12x^2y^2 + 8xy^3}{-4y}$

26. $\dfrac{-5m^3 + 10m^4 - 15m^5}{-5m^3}$

75. Division of polynomials by polynomials is like long division in arithmetic.

Example 1. — Divide 864 by 24.

Solution. — 1. $86 \div 24 = 3^+$.
2. $24 \times 3 = 72$; subtract . . . .
3. $14 \div 2 = 6^+$.
4. $24 \times 6 = 144$; subtract . . . .

$$\begin{array}{r|r} & 36 \\ \hline 24 & 864 \\ & 72 \\ \hline & 144 \\ & 144 \\ \hline \end{array}$$

Example 2. — Divide $10x^3 - 21x^2 - 11x + 12$ by $2x^2 - 3x - 4$.

Solution. —
1. $10x^3 \div 2x^2 = 5x$.
2. $(2x^2 - 3x - 4) \times (5x)$; subtract .
3. $(-6x^2) \div (2x^2) = -3$.
4. $(2x^2 - 3x - 4) \times (-3)$; subtract .

$$\begin{array}{r|l} & 5x \quad -3 \\ \hline 2x^2 - 3x - 4 & 10x^3 - 21x^2 - 11x + 12 \\ & 10x^3 - 15x^2 - 20x \\ \hline & \qquad\;\; -6x^2 + 9x + 12 \\ & \qquad\;\; -6x^2 + 9x + 12 \\ \hline \end{array}$$

The following explanation of the process may be given.

We are to find an expression which, when multiplied by the divisor, $2x^2 - 3x - 4$, will produce the dividend.

The term containing the highest power of x in the product is the product of the terms containing the highest powers of x in the multiplicand and multiplier.

Therefore, $10x^3$ is the product of $2x^2$ and the term containing the highest power of x in the quotient. Dividing $10x^3$ by $2x^2$ gives $5x$, which is the term containing the highest power of x in the quotient.

When the dividend is formed, the divisor is multiplied by each term of the quotient, and the results are added. Now reversing the process, multiply the divisor by $5x$ and subtract the result, $10x^3 - 15x^2 - 20x$, from the dividend.

The remainder, $-6x^2 + 9x + 12$, must be the product of the divisor and the rest of the quotient. Consider it a new dividend.

Its term containing the highest power of x, $-6x^2$, is the product of $2x^2$ and the term containing the next lower power of x in the quotient. Dividing $-6x^2$ by $2x^2$ gives the next term, -3. Multiply the divisor by -3 and subtract the result from the previous remainder. There is now no remainder.

The quotient is therefore $5x - 3$.

Rule. — To divide a polynomial by a polynomial:

1. Arrange the dividend and the divisor in either ascending or descending powers of some common letter.

2. Divide the first term of the dividend by the first term of the divisor, and write the result as the first term of the quotient.

3. Multiply the whole divisor by the first term of the quotient; write the product under the dividend and subtract it from the dividend.

4. Consider the remainder a new dividend, and repeat Steps 1, 2, and 3.

NOTE 1. — The terms of the quotient are placed above the terms of the dividend from which they are obtained.

NOTE 2. — Like terms are arranged carefully in a vertical column.

NOTE 3. — Spaces should be left for any powers of the common letter which are not present in the dividend.

NOTE 4. — As in arithmetic, there may be a final remainder.

Example 3. — Divide $x^2y^2 + x^4 - y^4$ by $-xy + y^2 + x^2$.

Solution. —

$$\begin{array}{r|rrrrr}
 & \multicolumn{5}{l}{x^2 + xy + y^2} \\ \cline{2-6}
x^2 - xy + y^2 & x^4 & & + x^2y^2 & & - y^4 \\
 & x^4 & - x^3y & + x^2y^2 & & \\ \cline{2-6}
 & & + x^3y & & & - y^4 \\
 & & + x^3y & - x^2y^2 & + xy^3 & \\ \cline{3-6}
 & & & x^2y^2 & - xy^3 & - y^4 \\
 & & & x^2y^2 & - xy^3 & + y^4 \\ \cline{4-6}
 & & & & & -2y^4
\end{array}$$

NOTE. — The quotient is $x^2 + xy + y^2$; the remainder is $-2y^4$. As in arithmetic, the complete quotient may be written:

Complete quotient: $x^2 + xy + y^2 + \dfrac{-2y^4}{x^2 - xy + y^2}$.

Check. — Let $x = 1$, and $y = 1$. Then, dividend $= 1$, and divisor $= 1$.

Quotient $= 1 + 1 + 1 + \dfrac{-2}{1 - 1 + 1} = 3 + \dfrac{-2}{1} = 3 - 2 = 1$.

Since $1 \times 1 = 1$, the quotient is correct.

Another check would be to multiply the divisor by the quotient and add the remainder; the sum should equal the dividend.

EXERCISE 54

Divide:

1. $x^2 + 5x + 6$ by $x + 3$
2. $x^2 + 8x + 12$ by $x + 2$
3. $y^2 + 8y + 15$ by $y + 5$
4. $m^2 + 13m + 12$ by $m + 1$
5. $A^2 + 10A + 24$ by $A + 6$
6. $r^2 - 18r + 32$ by $r - 2$
7. $s^2 - 12s + 35$ by $s - 7$
8. $t^2 + 72 - 17t$ by $t - 9$
9. $c^2 + 72 - 18c$ by $c - 6$
10. $d^2 + 13d + 36$ by $d + 4$
11. $x^2 + 2x - 8$ by $x - 2$
12. $t^2 + t - 30$ by $t + 6$
13. $a^2 - 6a - 27$ by $a - 9$
14. $m^2 + m - 20$ by $m + 5$
15. $n^4 - 5n^2 - 24$ by $n^2 + 3$
16. $x^2 - 16ax + 55a^2$ by $x - 5a$
17. $a^2 - 3ab - 108b^2$ by $a + 9b$
18. $x^2 - 2xz - 48z^2$ by $x - 8z$
19. $x^4 + 6x^2y - 27y^2$ by $x^2 - 3y$

20. $x^2y^2 - 18xy + 45$ by $xy - 3$

21. $15x^2 - 14x - 16$ by $3x + 2$

22. $6a^2 + 25 - 25a$ by $2a - 5$

23. $24a^2 - 38a + 15$ by $6a - 5$

24. $30x^2 + 7 - 47x$ by $6x - 1$

25. $28x^2 - 15y^2 + 23xy$ by $4x + 5y$

26. $20m^2 + 41mn + 20n^2$ by $5m + 4n$

27. $x^3 - 5x^2 - 17x + 66$ by $x - 6$

28. $6a^3 - 27a - a^2 + 20$ by $3a - 5$

29. $4x^3 - 5xy^2 + 4y^3 - 8x^2y$ by $2x - y$

30. $12a^3 + 17a^2b - 25b^3 - 20ab^2$ by $3a + 5b$

31. $x^4 - x^3y - 2x^2y^2 - 3xy^3 - y^4$ by $x^2 + xy + y^2$

32. $5n^3 + 8n^2 - 23n - 1$ by $5n^2 - 7n - 2$

33. $15x^3 - 30x - 8 - 19x^2$ by $-5x + 3x^2 - 4$

34. $2a^4 + 3a^2 - 8a^3 + 7 + 10a$ by $2a^2 - 7 - 4a$

35. $4m^3 - 3m - 15m^2 + 4$ by $m^2 - 3m - 3$

36. $2x^3 - 9x^2y + 17xy^2 - 12y^3$ by $2x - 3y$

37. $y^3 + 27$ by $y + 3$

38. $y^3 - 27$ by $y + 3$

39. $x^4 - y^4$ by $x - y$

40. $a^4 + b^4$ by $a + b$

41. $8a^3 - 1$ by $2a - 1$

42. $x^4 - 16y^4$ by $x - 2y$

43. $27m^3 + 64n^3$ by $3m + 4n$

44. $a^5 + b^5$ by $a + b$

45. $16n^4 - 81$ by $8n^3 + 12n^2 + 18n + 27$

46. $6x^4 - 7x^3 - 28x^2 + 8x - 21$ by $3x^2 - 5x - 7$

47. $x^4 + x^2y^2 + y^4$ by $x^2 + xy + y^2$

HISTORICAL NOTE. — Stifel (1486–1567) seems to have been one of the first to divide a polynomial by a polynomial. Sir Isaac Newton (1642–1727), in a book published in 1707, pointed out the advantage of arranging the dividend and divisor according to ascending or descending powers of the same letter.

NOTE. — Supplementary examples appear on page 431.

VII. SIMPLE EQUATIONS

76. An **identity** is an equation which is satisfied by all values of the literal numbers except possibly zero.

It is a declarative sentence. It is a statement of actual equality under all conditions.

Thus, $3\,x(a-b) = 3\,ax - 3\,bx$ is an identity.
If $a = 3$, $b = 1$, and $x = 2$, $3\,x(a-b) = 3 \cdot 2(3-1) = 6 \cdot 2$ or 12.
Also, $3\,ax - 3\,bx = 3 \cdot 3 \cdot 2 - 3 \cdot 1 \cdot 2 = 18 - 6$ or 12.

Try any other values of a, b, and x. You will find that they also satisfy this equation.

Similarly, $(3\,a + 2\,b) + (a - b)$ *is identically equal to* $4\,a - b$
$15\,x^2 - 3\,x^2$ *is identically equal to* $12\,x^2$
$7\,x(2\,y - z)$ *is identically equal to* $14\,xy - 7\,xz$
$30\,y^3 \div 3\,y$ *is identically equal to* $10\,y^2$

That is, *the result obtained from adding, subtracting, multiplying, or dividing two expressions is identically equal to the indicated combination of the numbers.* It is for this reason that you can check solutions of such problems by substitution of any numbers for the letters.

77. A **conditional equation** is an equation involving one or more literal numbers, which **is not satisfied by all values of the literal numbers.**

Thus: *a.* $x + 2 = 5$ is not satisfied by any value of x except $x = 3$.
b. $x^2 - 5\,x = -6$ is satisfied by $x = 2$ and by $x = 3$, but by no other values of x.

A conditional equation is like an interrogative sentence; it implies a question.

Thus, $3\,x - 5 = 4$, asks "for what value of x is $3\,x - 5 = 4$?"
The answer is, "x must be 3," for $3 \times 3 - 5 = 9 - 5 = 4$.

The word " equation " usually refers to a conditional equation.

78. If an equation has only one unknown number, if the unknown does not appear in the denominator of any fraction, and if the unknown appears only with the exponent 1, then the equation is called an **equation of the first degree,** or a **simple equation.**

Thus, $3x - 5 = 4$ is a simple equation.

Historical Note. — The idea of the degree of an equation was introduced by Descartes.

Properties of Equations

79. You have learned four rules to be used in solving simple equations. If you do not recall these rules, review §§ 11, 12, 43, and 47.

There are **three mechanical methods of solution** *which can be used,* although use of them is unnecessary.

80. Transposition.

Example. — Solve the equation $10x - 5 = 3x + 30$.

Solution. — 1. $10x - 5 = 3x + 30$.

2. A_5 $10x = 3x + 30 + 5$.

3. S_{3x} $10x - 3x = 30 + 5$.

4. C. T. $7x = 35$.

5. D_7 $x = 5$.

By adding 5 to both members of Step 1, -5 is made to disappear from the left member and $+5$ to appear in the right member in Step 2. *It is as if -5, with its sign changed, were taken from the left side and placed on the right side.*

In Step 3, subtracting $3x$ from both members of Step 2, causes $3x$ to disappear from the right side of Step 2, and $-3x$ to appear in the left side in Step 3. Again, *it is as if $3x$, with its sign changed, were moved from the right side to the left side.*

These are two examples of transposition ("place across").

Rule. — A term may be transposed from one member of an equation to the other, provided its sign is changed.

Historical Note. — Our word algebra, curiously, is associated with this process, *transposition.* About the first quarter of the ninth century an Arabian mathematician, Mohammed ben Musa, wrote an algebra, for the title of which he used *Ilm al-jabr wa'l muqabalah. Al-jabr* meant the process of transposing terms. This title was used in various forms in Europe until about the fifteenth century, when the last part was dropped and *algebra* came into use.

The Greeks had no special name for their algebra. The Hindu writers called it reckoning with unknowns.

81. Canceling terms in an equation.

Example. — Solve the equation $x + a = b + a$.

Solution. — 1. $x + a = b + a$.

2. S_a $x = b$.

Notice that the term $+ a$, which appeared in both members of the given equation, has disappeared from both members in Step 2.

The result in Step 2 is the same as if $+ a$ were simply dropped from both members. **It was not simply dropped**; *it was subtracted from both members.* However,

Rule. — If the same term, preceded by the same sign, occurs in both members of an equation, it may be canceled (dropped) from both members.

Example. — Solve the equation

$$5x + 9x - 8 + 3 = 9x + 18 - 8 + 2x.$$

Solution. — 1. $5x + \cancel{9}\cancel{x} - \cancel{8} + 3 = \cancel{9}\cancel{x} + 18 - \cancel{8} + 2x$.

2. Since $+ 9x$ appears in both members, it can be canceled; also $- 8$ can be canceled.

$$\therefore\ 5x + 3 = 18 + 2x.$$

3. Transposing
$+ 2x$ and $+ 3$, $5x - 2x = 18 - 3$.

4. C. T. $3x = 15$.

5. D_3 $x = 5$.

Check. — Substituting 5 for x *in the original equation,*

does $25 + 45 - 8 + 3 = 45 + 18 - 8 + 10$?

Does $73 - 8 = 73 - 8$? Yes.

Note. — First, cancel any terms which appear in both members; second, transpose terms containing unknowns to the left side, and terms containing knowns to the right side.

EXERCISE 55

Solve the following equations:

1. $4m - 2 = m + 7$

2. $8t + 5 = 3t + 35$

3. $7x - 10 = 50 - 5x$

4. $5y - 3 = 11 + 2y$

5. $7z - 11 = 15 - 6z$

6. $15r - 3 = 4 - 6r$

7. $13w - 92 + 8 = 4 + 5w + 8$

8. $12g - 5g + 1 = 8g + 3 - 5g$

9. $11a - 14 + a = 9a + 4 + a$

10. $20s - 19s + 3 = -19s + 13 - 5s$

11. $4x - 7 + 5x - 9 = 61 + 5x - 3x$

12. $32A + 16 + 14A + 1 - 9A = 7A + 16 + 14A + 13$

13. $17m + 16m - 5m - 4 = 5m + 8 - 3m + 2 + 16m$

14. $14t - 13 + 11t - 17 - 2t = 8t - 13 - 16t + 25 + 11t$

15. $8k + 6.5 - 48 - 73k = 48 + 5k + 6.5 - 17k - 73k$

82. Changing signs in an equation. Often we obtain an equation like

$$-x + 8 = +3.$$

We prefer that the sign of x be plus. To make it so, multiply both members by -1. Then,

$$x - 8 = -3.$$

The sign of each term now is exactly opposite to its sign in the original equation. *The result of multiplying by -1 is the same as if the signs were simply changed.*

Rule. — The signs of all terms of an equation may be changed, without destroying the equality.

83. Negative roots of equations.

Example. — Solve the equation

$$7 - 5x - 9x = 15 - 9x - 3x.$$

Solution. — 1. $7 - 5x - 9x = 15 - 9x - 3x.$

2. Canceling $-9x$, $7 - 5x = 15 - 3x.$

3. Transposing $+7$
and $-3x$, $3x - 5x = 15 - 7.$

4. C. T. $-2x = 8.$

5. D_{-2} $x = -4.$

Check. — Substitute -4 for x in the first equation.

Does $7 - 5(-4) - 9(-4) = 15 - 9(-4) - 3(-4)$?

Does $7 + 20 + 36 = 15 + 36 + 12$?

Does $63 = 63$? Yes.

NOTE. — In Step 5, $(-2x) \div (-2) = x$; $(+8) \div (-2) = -4$. Or, you may multiply first by -1 in Step 4, and then divide by $+2$.

EXERCISE 56

Solve the following equations either by the rules taught in §§ 11, 12, 43, and 47, or by the ones taught in §§ 80, 81, and 82.

1. $2 - 3x = x - 8$

2. $4m - 5 = 6m + 1$

3. $13 - 6x = 3 - 11x$

4. $7t + 10 = 19t - 2$

5. $12 - 5n = n + 36$

6. $11 + 7p = 3p - 13$

7. $9r - 5 = 4r + 5$

8. $23 - 7z = -15z + 15$

9. $16 + 3w = -2w + 6$

10. $19 - 16y = 24y - 1$

11. $3(m + 1) - 9 = 5(2m + 7) - 6$ (Review § 64, page 83.)

12. $8t - 5(4t + 2) = 3(5 - 3t) - 1$

13. $8 - 3x(x - 4) = 2 - x(3x - 15)$

14. $6(3 - 2c) - 5c = 5c + 7(4 - 3c)$

15. $3(3x + 7) - 4(6x - 3) = 10 - 6(3x - 4)$

16. $10r - (4r + 3) = 3(8r - 2) + 4(1 - 4r)$

17. $m(m - 7) - 20 = 2m(m - 11) - m(m - 13)$

18. $6(4x - 3) = 5 - 10(3 - x)$

19. $4(3r - 7) + 1 = 5(6r - 1) - 2(5 - 9r)$

20. $19 - 6(4c + 1) = 5(8c + 1) - 12(4c - 1)$

84. Solution of problems. You have been taught how to form the equations for all the problems of the following list, excepting one kind. Example 1 is one of this new kind. In it, you must express *the relation between* a *dividend*, a *divisor*, the *quotient*, and the *remainder*.

If 19 is divided by 7, the quotient is 2 and the remainder is 5.
Then $19 = 2 \cdot 7 + 5$.
That is, **dividend = quotient × divisor + remainder.**

EXERCISE 57

1. *a.* Write the equation expressing the relation between dividend, divisor, quotient, and remainder when 46 is divided by 8.

b. Select a dividend and a divisor, and write the equation connecting your dividend, divisor, and the quotient and remainder you obtain.

2. If $(37 - x)$ is divided by x, the quotient is 3 and the remainder is 5. Form the equation expressing this relation, and solve it for x.

3. If n is divided by $65 - n$, the quotient is 3 and the remainder is 5. Find n.

4. Separate 54 into two parts such that the larger part divided by the smaller part gives 3 as quotient and 10 as remainder.

Hint. — What is the dividend in this example? The divisor?

5. Separate 125 into two parts such that the larger divided by the smaller gives 3 as quotient and 21 as remainder.

6. If 13 be added to a certain number, and the sum be multiplied by 2, the product is 60. What is the number?

7. Separate 28 into two parts such that 5 times the smaller exceeds 4 times the larger by 5.

8. Separate 43 into two parts such that 2 times the larger, diminished by 4 times the smaller, equals 7 less than the larger.

9. There are three consecutive integers such that 4 times the first, diminished by the second, equals 12 more than 2 times the third. What are they?

10. What are the three sides of the triangle whose perimeter is 121 in., if the second side is twice the first, and the third side is 19 in. shorter than the second?

11. The Lincoln Memorial at Washington stands on a platform whose perimeter is 676 ft. The length of the platform exceeds its width by 70 ft. What are the dimensions of the platform?

12. The foundation of the Lincoln Memorial goes down to the rock below the level of the Potomac. From the rock below grade to the top of the memorial is 172 ft. If the height above ground be divided by the depth below, the quotient is 2 and the remainder is 22. What is the depth below grade, and what is the height above grade?

13. The age of A is now 4 times the age of B. Four years from now A's age will be one year more than 3 times B's age then. What are their present ages?

14. John is 4 years older than Charles. Eight years ago, twice Charles' age exceeded John's age then by 1 year. How old is each now?

15. The treasurer of a club, after collecting dues, had $17.00 in nickels, dimes, quarters, and half-dollars. The number of dimes was 3 times the number of nickels; the number of quarters exceeded the number of dimes by 2; the number of half-dollars was 3 less than 5 times the number of nickels. How many of each coin did he have?

16. The total angle around a point (see Example 20, page 24) consists of certain four angles. The second angle is twice the first; the third exceeds 3 times the first by 10°; the fourth exceeds the sum of the second and third by 10°. How large is each of these angles?

17. The second angle of a certain triangle exceeds the first by 30°; the third angle exceeds the second by 15°. How large are the angles?

18. The elevation of Mt. Whitney, in California, the highest point recorded in the United States, is 14,501 feet, measured from sea level. The lowest point of dry land in the United States is in Death Valley, California. If 52 times the elevation of Death Valley be diminished by 45, and the result be increased by the elevation of Mt. Whitney, the sum is zero. Find and interpret the elevation of Death Valley.

19. There are three consecutive even integers such that the sum of 3 times the first, four times the second, and 3 times the third is 200. What are the integers?

20. How many pounds of 50¢ tea and how many pounds of 90¢ tea must be taken to make a mixture of 100 lb. of tea worth 60¢ per pound?

EXERCISE 58

Interest Problems

1. What is the simple interest on $500 at 6% for 1 year? For 3 years? For n years?

2. What is the simple interest on P dollars at 6% for 1 year? For 4 years? For t years?

3. What is the simple interest on $300 at $r\%$ for 1 year? For 3 years? For t years?

4. What is the simple interest on P dollars at $r\%$ for 1 year? For 5 years? For t years?

5. The total income from two investments is $135. Part of the money is invested at 6% and part at 7%. The total amount invested is $2000. Find the two sums invested.

Solution. — 1. Let s = the number of dollars invested at 6%.
2. $\therefore\ 2000 - s$ = the number of dollars invested at 7%.
3. $\therefore\ .06\,s$ = the interest on the first sum,

4. and $.07(2000 - s) =$ the interest on the second sum.

5. $\therefore\ .06\,s + .07(2000 - s) = 135.$

(Complete this solution and check it.)

NOTE. — The essential steps may be charted as below:

	NO. OF DOLLARS	RATE OF INTEREST	ANNUAL INTEREST
First sum	s	6%	$.06\,s$
Second sum	$(2000 - s)$	7%	$.07(2000 - s)$

$$\therefore\ .06\,s + .07(2000 - s) = 135.$$

6. One sum is invested at 6% and a second sum at 8%. The total income is $228. The second sum is 4 times as large as the first. What are the sums?

7. A man has one sum invested at 5%, and a second sum, $1500 larger than the first, invested at 7%. His total income from these sums is $321. How much has he invested at each rate?

8. A man has $2500 invested at 5%. How much must he invest at 8% *to make his total income 6% of the total amount invested?*

9. If $1500 is invested at 5%, and $2000 at 6%, how much must be invested at 7% to make the total income 6% of the total amount invested?

10. The total of certain two investments is $2500, — one sum at 5%, and one at 6%. The annual interest from the former is $29 less than that from the latter. How much is invested at each rate?

85. Distance, rate, and time problems.

If a train travels 40 miles per hour for 6 hours, it goes 240 miles.

Observe the **rate** (r), 40 miles per hour; the **time** (t), 6 hours; the **distance** (d), 240 miles.

The distance is expressed as a number of units of length; as feet, rods, miles.

The distance equals the rate multiplied by the time. ($d = rt$)

In the illustration above, $240 = 40 \times 6$.

The time is expressed as a number of units of time; as minutes, hours, days, etc.

The time equals the distance divided by the rate. ($t = d \div r$)

In the illustration above, $6 = 240 \div 40$.

The rate is expressed as a number of units of length in the unit of time.

The rate equals the total distance divided by the time.

($r = d \div t$)

In the illustration above, $40 = 240 \div 6$.

EXERCISE 59

1. How far does a train go in 9 hours if the rate is:

a. 25 m. per hr.? *b.* r m. per hr.? *c.* $(x + 2)$ m. per hr.?

2. How far does a train go in h hours if the rate is:

a. 35 m. per hr.? *b.* x m. per hr.? *c.* $(r - 3)$ m. per hr.?

3. Give the rule for finding the distance when the rate and the time are known.

4. How long does it take an automobile to go 150 miles if the rate is:

a. 20 m. per hr.? *c.* 50 m. per d.?

b. n m. per hr.? *d.* $(r + 5)$ m. per d.?

5. How long does it take to go N miles if the rate is:

a. 30 m. per hr.? *b.* $(r + 3)$ m. per hr.?

6. Give the rule for finding the time when the rate and the distance are known.

7. At what rate is a man traveling who goes 300 miles in:

a. 10 hr.? *b.* h hr.? *c.* $(x + 4)$ hr.?

8. What is the rate if an object moves D miles in:

a. 12 hr.? *b.* 3 d.? *c.* t hr.?

9. Give the rule for finding the rate when the distance and the time are known.

10. One man travels 25 miles per hour for x hours, and a second man travels 20 miles per hour for 2 hours less time.

a. How far does the first man travel?

b. How long does the second man travel?

c. How far does the second man travel?

d. Suppose the sum of the distances traveled by the two men is 100 miles. Write the equation expressing this fact.

e. Solve the equation, finding the time x, and check the solution by finding the distances in parts *a* and *c* and comparing their sum with 100.

11. A and B travel toward each other from points which are 300 miles apart, — A at the rate of 20 miles per hour, and B at the rate of 25 miles per hour. In how many hours will they meet, if they start at the same time?

Solution. — 1. Since they start at the same time and travel until they meet, they travel the same number of hours.

Let h = the number of hours they travel.

2.

Then for	the time is	the rate is	the distance is
one man	h hours	20 m. per hr.	$20h$ m.
the other man	h hours	25 m. per hr.	$25h$ m.

3. $\therefore 20h + 25h = 300.$ Since they started 300 m. apart and travel until they meet.

(Complete and check the solution.)

12. Two men who are traveling in opposite directions at the rates of 18 and 22 miles per hour respectively, started at the same time from the same place. In how many hours will they be 250 miles apart?

13. A passenger and a freight train, whose rates are 30 miles and 18 miles per hour respectively, start toward each other at the same time from points which are 168 miles apart. How soon will they meet and how far will each have traveled?

14. A freight and a passenger train started towards each other at the same time from points which are 225 miles apart. If the passenger train's rate was 30 miles per hour, and they meet in 5 hr., what was the rate of the freight train?

15. A and B started towards each other at the same time from points which are 240 miles apart, and met in 5 hours. If A traveled 12 miles per hour more than B, what were their rates?

16. Repeat Example 15, assuming A's rate was 10 miles more per hour than B's rate.

17. A freight and a passenger train started in opposite directions from the same place at the same time. Assume that the passenger train traveled twice as rapidly as the freight train. What were their rates if they were 300 miles apart in 5 hours?

18. Two automobiles started at the same time from the same point, in opposite directions. The first traveled 5 miles more per hour than the second. At the end of 8 hours they were 360 miles apart. What were their rates?

19. A and B started toward each other at the same time from points which are 252 miles apart. A traveled 4 miles more per hour than B. How fast did each travel if they met in 7 hours?

20. An automobile party is traveling 15 miles per hour. At what rate must a second party travel in order to overtake the first in 3 hours if it starts from the same place 1 hour after the first?

Hint. — If the second overtakes the first, how do their "distances" compare? Which party traveled the longer time and how much?

21. One automobile party is traveling 15 miles per hour. In how many hours will a second party, traveling 25 miles per hour, overtake the first, if the second party starts from the same place 2 hours after the first party left?

22. A left a certain point at 7 A.M., traveling 20 miles per hour. At 9 A.M., B started after him. How rapidly did B travel if he overtook A in 4 hours?

23. A left a certain point at 8 A.M. At 9 A.M., B started after him at the rate of 30 miles per hour, and overtook him in 3 hours. How fast had A traveled?

24. A and B, traveling at 18 miles and 22 miles per hour respectively, started from the same place in opposite directions.

a. A started at 6 A.M. and B at 8 A.M. How long had each traveled when they were finally 256 miles apart?

b. Solve the problem if B started at 6 A.M. and A at 8 A.M.

25. A and B, whose rates are 20 miles and 25 miles per hour respectively, travel toward each other from points which are 300 miles apart.

a. How long and how far will each have traveled when they meet if A starts at 6 A.M. and B at 9 A.M.?

b. Solve the same problem if B starts at 6 A.M. and A at 9 A.M.

26. Chicago, and Madison, Wisconsin, are about 140 miles apart. If a train starts from Chicago for Madison at 1 P.M., traveling 35 miles per hour, and one starts from Madison for Chicago, at 3 P.M., traveling 30 miles per hour, at what time will they meet?

SUPPLEMENTARY TOPICS

86. Further exercise in the use of formulæ.

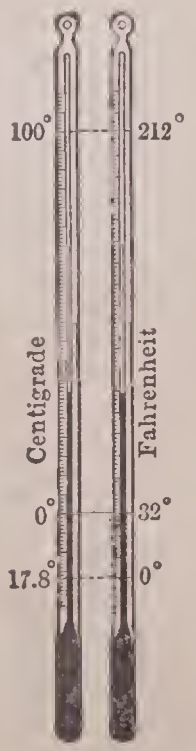
***a.* Relation between Fahrenheit and Centigrade temperature readings.**

You are familiar with the Fahrenheit thermometer. Scientists and people in many foreign countries use the Centigrade thermometer.

On the Fahrenheit thermometer, the freezing point is + 32° and the boiling point is + 212°; *between, there are* 180 *Fahrenheit degrees.* On the Centigrade thermometer, the freezing point is 0° and the boiling point is 100°; *between, there are* 100 *Centigrade degrees.*

∴ 100 Centigrade degrees = 180 Fahrenheit degrees

or 1 Centigrade degree = $\frac{9}{5}$ Fahrenheit degree.



Example. — What Fahrenheit reading corresponds to $+15°$ C.?

Solution. — 1. 15 Centigrade degrees $= 15(\frac{9}{5})$ or 27 Fahrenheit degrees.

2. $+15°$ C. means 15 Centigrade degrees above freezing.

3. $\therefore$ the corresponding Fahrenheit reading must be 27 Fahrenheit degrees above freezing, or 27° above 32°. This is $+59°$ F.

4. $\therefore\ +15°\text{ C.} = +59°\text{ F.}$

EXERCISE 60

Find the Fahrenheit reading corresponding to:

1. $+20°$ C. **2.** $+35°$ C. **3.** $-5°$ C. **4.** $-25°$ C.

5. Derivation of a formula connecting Centigrade and Fahrenheit readings.

Solution. — 1. Let $C =$ the Centigrade reading;
and $F =$ the corresponding Fahrenheit reading.

2. C Centigrade degrees $= (\frac{9}{5}C)$ Fahrenheit degrees.

3. But C° is counted from 0, the Centigrade freezing point, and $(\frac{9}{5}C)$ must be counted from 32, the Fahrenheit freezing point.

$$\therefore\ F = 32 + \frac{9}{5}C.$$

Check. — 1. Let $C = 0$. $\therefore\ F = 32 + \frac{9}{5}\cdot 0$, or 32.
$\therefore$ the freezing points correspond.

2. Let $C = 100$. $\therefore\ F = 32 + \frac{9}{5}(100)$, or 212.
$\therefore$ the boiling points correspond.

NOTE. — Memorize this formula. It is used as follows:

Example 6. — Change 10° F. to the corresponding Centigrade reading.

Solution. — 1. The formula is $F = 32 + \frac{9}{5}C$.
$F = +10;\quad C = ?$

2. $\therefore\quad 10 = 32 + \frac{9}{5}C.$

3. $M_5\quad 50 = 160 + 9C.$

(Complete the solution, and tell what the result means.)

Find

7. F when $C = +40$

8. F when $C = -20$

9. C when $F = 0$

10. C when $F = +60$

11. F when $C = +60$

12. C when $F = -20$

13. Air liquefies when its temperature is reduced to − 182° C. What Fahrenheit temperature is this?

14. Paraffine melts at + 55° C. What is the corresponding Fahrenheit temperature?

15. Turpentine boils at 320° F., and alcohol at 172.4° F. What are the Centigrade temperatures that correspond?

16. The following record-breaking low temperatures have been produced. What are the corresponding Fahrenheit temperatures?

− 17° C., by Fahrenheit in 1714.
− 102° C., by Faraday in 1823.
− 262° C., by Dewar in 1898.
− 269° C., by Onnes in 1908.

b. **Miscellaneous formulæ.**

EXERCISE 61

1. $S = \frac{n}{2}\{2a + (n-1)d\}$ is an important formula in mathematics.

a. Find S when $n = 20$, $a = -5$, and $d = -2$.
b. Find d when $S = 165$, $n = 10$, and $a = 6$.
c. Find a when $S = 174$, $n = 12$, and $d = 3$.
d. Find a when $S = -160$, $n = 16$, and $d = -7$.

2. $A = P(1 + rt)$.

a. Find t when $A = \$3600$, $P = \$3000$, and $r = 5\%$.
b. Find r when $A = \$3025$, $P = \$2500$, and $t = 3\frac{1}{2}$.

3. $V = \frac{h}{6}(a + b + 4m)$.

a. Find V when $h = 10$, $a = 12$, $b = 10$, and $m = 11$.
b. Find m when $V = 126$, $a = 13$, $h = 9$, and $b = 15$.

4. $c^2 = a^2 + b^2 - 2ap$.

a. Find p when $c = 14$, $b = 12$, and $a = 9$.
b. Find p when $c = 15$, $b = 12$, and $a = 9$.
c. Find p when $c = 16$, $b = 12$, and $a = 9$.

VIII. SPECIAL PRODUCTS AND FACTORING

87. In arithmetic, it is found necessary to memorize the multiplication table as an aid in multiplication, division, and factoring.

Thus, you know now that $6 \cdot 7 = 42$; that $42 \div 7 = 6$; and that $42 = 6 \cdot 7$.

In algebra, also, certain forms of expressions occur frequently, which must be multiplied, divided, or factored mentally.

88. To factor an algebraic expression is to find two or more expressions which will produce the given expression when they are multiplied together.

Review the definitions of factor (§ 33) and common factor (§ 15).

89. A number (or expression) which has no factors except itself and unity is called a **prime number**; as, 3, a, and $x + y$.

A monomial is expressed in terms of its prime factors thus:

$$12\,a^3b^2c = 2 \cdot 2 \cdot 3 \cdot a \cdot a \cdot a \cdot b \cdot b \cdot c.$$

90. Squaring a monomial.

Development. — 1. What does x^2 mean? $(xy)^2$? $(2\,r^3s)^2$?

2. Find each of the following squares by multiplication:

a. $(2\,xy)^2$; *b.* $(3\,a^2b^2)^2$; *c.* $(-2\,r^3s^4)^2$.

3. Compare the exponent of each letter of the square with the exponent of that letter in the given monomial.

Rule. — To square a monomial:

Square its numerical coefficient, and multiply the result by each of the literal factors of the monomial, giving each letter twice its original exponent.

Example. — Find $(-5\,x^2y^3)^2$.

Solution. — $(-5\,x^2y^3)^2 = 25\,x^4y^6$.

EXERCISE 62

1. What sign does the square of any number have?

2. Learn thoroughly the squares of the integers from 1 to 20.

Give at sight:

3. $(a^3b)^2$
4. $(-a^2b^3)^2$
5. $(3x^2)^2$
6. $(ab^2c)^2$
7. $(-5m)^2$
8. $(-4xy)^2$
9. $(-8a^2b)^2$
10. $(+9m^3b^2)^2$
11. $(-11r^2st^2)^2$
12. $(14mxy^3)^2$
13. $(-12a^3b^4)^2$
14. $(-15x^2y)^2$
15. $(\frac{1}{2}a^2)^2$
16. $(-\frac{2}{3}mn)^2$
17. $(-\frac{5}{8}ab)^2$
18. $\left(\frac{9x}{11y}\right)^2$
19. $\left(-\frac{12t^3}{13}\right)^2$
20. $\left(\frac{14m}{15x^2}\right)^2$

91. The square root of a monomial. If an expression can be resolved into two equal factors, it is said to be a perfect square, and one of the factors is said to be its **square root.**

Thus, $4a^2b^6$ is equal to $2ab^3 \times 2ab^3$; hence it is a perfect square and $2ab^3$ is its square root.

NOTE. — $4a^2b^6$ is also equal to $(-2ab^3) \times (-2ab^3)$, so that $-2ab^3$ *is also a square root. In this chapter, only the positive square root will be considered. It is called the principal square root.*

The following questions lead to the rule for extracting the square root of a perfect square monomial.

Development. — 1. What sign does the square of any monomial have?

2. When squaring a monomial, what do you do with the exponents of the literal factors? with the coefficient?

3. In finding the square root, then, what should you do with the exponents of the literal factors? with the coefficient?

4. Find the square root of each of the following monomials, and test the result by multiplication:

a. x^4; *b.* $4x^2y^2$; *c.* $16r^2s^4$; *d.* $25x^2y^2z^6$.

Rule. — 1. A perfect square monomial is positive, has a perfect square numerical coefficient, and only even numbers as exponents.

2. To find its square root: find the square root of its numerical coefficient, and multiply the result by the literal factors of the monomial, giving each letter one half its original exponent.

The symbol for extracting the square root is the radical sign, $\surd$; the vinculum is usually combined with it, $\sqrt{}$

Example. — Find the square root of $25\,m^4n^6$.

Solution. — $\sqrt{25\,m^4n^6} = 5\,m^2n^3$.

EXERCISE 63

Find the indicated square root:

1. $\sqrt{4\,x^2}$
2. $\sqrt{16\,y^4}$
3. $\sqrt{9\,m^6}$
4. $\sqrt{25\,a^2b^4}$
5. $\sqrt{64\,r^2s^6}$
6. $\sqrt{81\,c^4d^2}$
7. $\sqrt{\dfrac{4\,a^2}{9\,b^2}}$
8. $\sqrt{100\,m^4}$
9. $\sqrt{169\,a^2b^2}$
10. $\sqrt{225\,c^6}$
11. $\sqrt{196\,x^4}$
12. $\sqrt{256\,y^2}$
13. $\sqrt{49\,t^6}$
14. $\sqrt{\frac{1}{9}\,c^2d^2}$
15. $\sqrt{\frac{4}{9}\,m^2}$
16. $\sqrt{\frac{16}{25}\,a^2b^2}$
17. $\sqrt{\frac{81}{169}\,x^4}$
18. $\sqrt{256\,m^2n^4}$
19. $\sqrt{\frac{196}{121}\,a^4b^6}$
20. $\sqrt{\frac{100}{81}\,c^4d^2}$
21. $\sqrt{\dfrac{169\,x^4}{196\,y^2}}$
22. $\sqrt{\dfrac{400\,a^6}{121\,b^4}}$
23. $\sqrt{\dfrac{144\,n^4}{225\,m^2}}$
24. $\sqrt{\dfrac{900\,c^8}{169\,d^2}}$

FACTORING POLYNOMIALS

92. It is not always possible to factor a polynomial. Those polynomials which can be factored are the products of certain special forms of number expressions.

Note. — In this chapter, you will be taught some elementary methods of factoring. In Chapter XV, other methods are taught.

CASE I

Type Form: $a(b + c) = ab + ac$

93. **Multiplying a polynomial by a monomial is taught in § 62.**

EXERCISE 64

Find mentally:

1. $2a(x - y + z)$
2. $-3x(x^2 + xy - y^2)$
3. $+4cd(c^2 - cd + d^2)$
4. $-5m^2(x^3 - y^3)$
5. $-6ab(a^2 - 2ab + b^2)$
6. $-\frac{1}{2}x(4x^2 - 6x - 8)$
7. $\frac{1}{3}mn(12m^3 - 3m^2n - 6n^3)$
8. $-a(c - d + e)$
9. $\frac{2}{3}t(r + s - t)$
10. $\frac{1}{2}r(a + b + c + d)$

94. **Factoring a polynomial whose terms have a common monomial factor.**

Example. — Factor $14xy^3 - 70x^3y^2$.

Solution. — 1. 14 and 70 can both be divided by 2, 7, and 14.
14 is the greatest common factor of 14 and 70.
We select it as part of the monomial factor.

2. x is a factor of both $14xy^3$ and $70x^3y^2$. It is the highest power of x which is a common factor of $14xy^3$ and $70x^3y^2$.
We select x as part of the monomial factor.

3. y^3 appears in $14xy^3$, but only y^2 appears in $70x^3y^2$. Then y^2 is the highest power of y which is a factor of both $14xy^3$ and $70x^3y^2$.
We select y^2 as part of the monomial factor.

4. $\therefore$ the monomial factor is $14xy^2$.

5. $14xy^3 \div 14xy^2 = y$; $-70x^3y^2 \div 14xy^2 = -5x^2$.

6. $\therefore 14xy^3 - 70x^3y^2 = 14xy^2(y - 5x^2)$.

Rule. — To factor a polynomial whose terms have a common monomial factor:

1. Find the greatest common factor of its terms.

2. Divide the polynomial by it.

3. The factors are the common factor found in Step 1, and the quotient obtained in Step 2.

EXERCISE 65

Factor the following polynomials:

1. $5a + 5b$

2. $7t - 7r$

3. $3s - 12x$

4. $mx - my$

5. $3ax^2 - 3ay^2$

6. $4rm^2 - 16ry^2$

7. $\pi r^2 + \pi rl$

8. $P + Prt$

9. $\frac{1}{2}na + \frac{1}{2}nl$

10. $2\pi r^2 + 2\pi rh$

11. $3x^2 - 6xy + 3y^2$

12. $ma^2 + 6ma + 9m$

13. $tx^2 - 4txy + 4ty^2$

14. $3x^2 + 3xy + 3y^2$

15. $2r^2x^2 - 4r^2x + 2r^2$

16. $6x^2 - 12x^3 - 18x^4$

17. $an^2 - 7an - 18a$

18. $\frac{1}{3}rB + \frac{1}{3}sB + \frac{1}{3}tB$

19. $\frac{1}{6}ha + \frac{1}{6}hb + \frac{4}{6}hm$

20. $9m^2x - 6mx - 15x$

21. $7y^2z + 21yz - 126z$

22. $3ab^2 - 15ab + 18a$

23. $2m^3n^2 + 4m^3n - 126m^3$

24. $4m^2r^2 + 8m^2r - 60m^2$

25. $24m^2a + 18mna - 15n^2a$

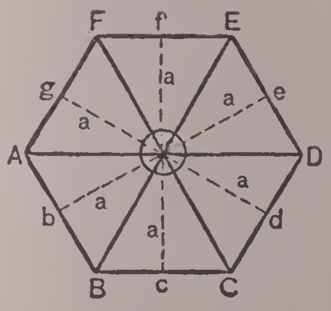
26. $9c^3x^2 - 4c^3xy - 13c^3y^2$

27. $3c^2x - 132cx - 135x$

28. $3x^3y^2 + 6x^2y^2 + xy^2 + 2y^2$

29. $\frac{1}{2}ax + \frac{1}{2}am + \frac{1}{2}az - \frac{1}{2}at$

30. $\frac{1}{3}\pi r^2h + \frac{1}{3}\pi R^2h + \frac{1}{3}\pi rRh$



31. Suppose that the polygon $ABCDEF$ can be divided into six triangles, such that their altitudes are equal. Call the altitudes each a, and the bases b, c, d, e, f, and g.

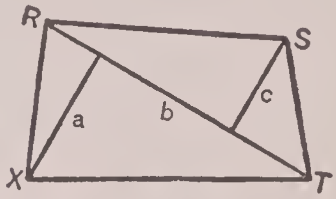
a. What is the area of $\triangle OBC$? $\triangle ODC$? $\triangle ODE$? etc.

b. Indicate the sum of these areas.

c. Simplify that sum by removing the monomial factor.

d. Simplify the result by substituting p for $(b + c + d + e + f + g)$.

32. Suppose that the altitude of $\triangle RXT$ is a, and the altitude of $\triangle RST$ is c. The base of each is b.

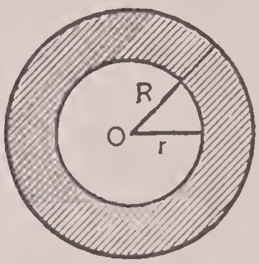


a. Represent the area of each.

b. Indicate the sum of these areas.

c. Simplify the result by removing the monomial factor.

33. The area of a circle whose radius is r is πr^2.

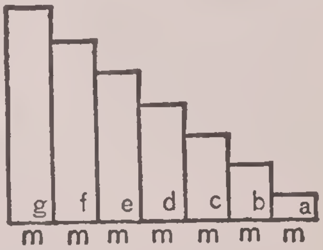


a. What is the area of the circle of radius R?

b. How can you find the area of the ring between the large and small circles? Indicate this area.

c. Simplify the result of Step *b* by removing the monomial factor.

34. Suppose that, in the adjoining figure, the rectangles have equal bases of length m, and altitudes of length a, b, c, etc.



a. Represent the area of each.

b. Indicate the sum of these areas.

c. Simplify the resulting expression by removing the monomial factor.

CASE II

Type Form: $(a + b)(a - b) = a^2 - b^2$

95. The product of the sum and the difference of two numbers.

NOTE. — By "difference of" two numbers, we mean *the first minus the second.*

Development. — 1. Find by multiplication as in § 63 the following products, and write the results as in part *a*:

a. $(x + 3)(x - 3) = x^2 - 9$

b. $(m + 7)(m - 7) = ?$

c. $(k + 10)(k - 10) = ?$

d. $(r + 9)(r - 9) = ?$

2. Observe the results in Step 1; try to find the following products mentally. Check the results by multiplication.

a. $(a + 6)(a - 6) = ?$ *c.* $(d + 4)(d - 4) = ?$
b. $(c + 8)(c - 8) = ?$ *d.* $(y + 5)(y - 5) = ?$

3. Find by multiplication $(a + b)(a - b)$.

Rule. — To find the product of the sum and the difference of two numbers:

1. **Square each of the numbers.**
2. **Subtract the second square from the first.**

Example 1. — Find $(5a^2 + m)(5a^2 - m)$.

Solution. — $(5a^2 + m)(5a^2 - m) = (5a^2)^2 - (m)^2 = 25a^4 - m^2$.

NOTE. — When doing such multiplication, do not write $(5a^2)^2 - (m)^2$. Do that part mentally and give at once the result $25a^4 - m^2$.

This is called *finding the result by inspection.*

Example 2. — Find mentally the product of 24 and 16.

Solution. — $24 \times 16 = (20 + 4)(20 - 4) = 400 - 16 = 384$.

(All this is to be done mentally.)

EXERCISE 66

Find by inspection:

1. $(x + 3)(x - 3)$
2. $(m + 5)(m - 5)$
3. $(t^2 - 2)(t^2 + 2)$
4. $(x^2 - 6)(x^2 + 6)$
5. $(m^3 + 4)(m^3 - 4)$
6. $(a + 2b)(a - 2b)$
7. $(r^2 + 3s)(r^2 - 3s)$
8. $(3x - b)(3x + b)$
9. $(7m - 4)(7m + 4)$
10. $(5x^2 - y)(5x^2 + y)$
11. $(10z + w^3)(10z - w^3)$
12. $(4m - .5n)(4m + .5n)$
13. $(6x^2 - 7y)(6x^2 + 7y)$
14. $(ab - 3c)(ab + 3c)$
15. $(a^5 - b^3)(a^5 + b^3)$
16. $(x^2 + y^2)(x^2 - y^2)$
17. $(a^4 + b^4)(a^4 - b^4)$
18. $(a + b)(a - b)(a^2 + b^2)$
19. $(x^3 + y^3)(x^3 - y^3)(x^6 + y^6)$
20. $(3xy^2 - 5z^3)(3xy^2 + 5z^3)$
21. $(\frac{1}{2}x - \frac{1}{3}y)(\frac{1}{2}x + \frac{1}{3}y)$
22. $(\frac{2}{3}m - \frac{1}{5})(\frac{2}{3}m + \frac{1}{5})$

23. $(\frac{3}{4}t^2 - 7)(\frac{3}{4}t^2 + 7)$

24. $(\frac{4}{9}x - \frac{3}{11})(\frac{4}{9}x + \frac{3}{11})$

25. $(\frac{3}{8}a^2 - \frac{2}{5}b)(\frac{3}{8}{}^2 a + \frac{2}{5}b)$

26. $(12 - 3)(12 + 3)$

27. $(20 + 2)(20 - 2)$

$$\left.\begin{array}{l}\textbf{28. } (25)(15)\\ \textbf{29. } (33)(27)\\ \textbf{30. } (42)(38)\end{array}\right\} \begin{array}{c}\text{See}\\ \text{Example 2}\\ \text{on page 124.}\end{array}$$

31. $17 \cdot 23$

32. $41 \cdot 39$

33. $53 \cdot 47$

34. $34 \cdot 26$

35. $44 \cdot 36$

36. $31 \cdot 29$

37. $28 \cdot 32$

38. $37 \cdot 43$

39. What is the cost of 18 yards of gingham at 22¢ per yard?

40. What is the value of 26 dozen of eggs at 34¢ per dozen?

96. Factoring the difference of two perfect squares.

Development. — 1. What is the product of $(a + 2)$ and $(a - 2)$? What, then, are the factors of $a^2 - 4$?

2. Find the factors of:

a. $a^2 - 9$ *b.* $m^2 - 16$ *c.* $k^2 - l^2$ *d.* $9r^2 - 4s^2$

3. What are the factors of $x^2 - y^2$?

Rule. — To factor the difference of two squares:

1. Find the square roots of the two perfect square terms.

2. One factor is the sum of the results; the other factor is the difference of the results.

Example 1. — Find the factors of $25r^4 - 16t^6$.

Solution. — 1. $25r^4 - 16t^6 = (5r^2)^2 - (4t^3)^2$.

2. $\therefore 25r^4 - 16t^6 = (5r^2 + 4t^3)(5r^2 - 4t^3)$.

NOTE. — Examine the given expression carefully to make certain it is the difference of two squares, as in Step 1.

However, when giving the factors, omit Step 1, and at once write what appears in Step 2.

Example 2. — Find mentally the value of $13^2 - 7^2$.

Solution. —
$$\begin{aligned}13^2 - 7^2 &= (13 + 7)(13 - 7)\\ &= 20 \times 6 = 120.\end{aligned}$$

Example 3. — Find mentally $(x^2 - y^2) \div (x - y)$.

Solution. — 1. Since $x^2 - y^2 = (x - y)(x + y)$,

2. $\therefore (x^2 - y^2) \div (x - y) = (x + y)$.

EXERCISE 67

Factor the following *when possible:*

1. $x^2 - 64$
2. $y^2 - 144$
3. $100 - z^2$
4. $x^4 - y^2$
5. $9\,m^2 - 1$
6. $x^4 - y^5$
7. $x^2 - 16\,y^2$
8. $a^2b^2 - 25$
9. $36\,t^6 - 1$
10. $81\,m^2 - a^4$
11. $121\,a^6 - 16\,b^2$
12. $49 - y^6$
13. $196 - a^2$
14. $256\,c^2 - d^2$
15. $16\,x^2 - 50\,y^2$
16. $x^2 - \frac{1}{4}$
17. $9\,c^2 - \frac{1}{16}\,d^2$
18. $\frac{1}{25}\,m^2 - 1$
19. $\frac{4}{9}\,x^2 - 49$
20. $144\,x^8 - y^6$
21. $\frac{9}{25}\,x^{10} - 1$
22. $196\,a^2 - b^4$
23. $64\,x^2y^4 - z^6$
24. $81\,t^{10} - 169\,m^4$
25. $100\,a^2b^3 - c^4$
26. $\frac{25}{49}\,a^2 - \frac{36}{121}\,b^2$
27. $225\,m^2 - 256\,n^2$
28. $\frac{36}{x^2} - \frac{169}{y^2}$
29. $\frac{a^4}{25} - \frac{b^4}{49}$
30. $\frac{16\,x^2}{y^2} - \frac{m^2}{25}$

Find the values of the following as in Example 2, page 125, doing all mentally if possible.

31. $17^2 - 8^2$
32. $14^2 - 11^2$
33. $22^2 - 8^2$
34. $34^2 - 6^2$
35. $26^2 - 14^2$
36. $21^2 - 19^2$
37. $33^2 - 27^2$
38. $46^2 - 40^2$
39. $27^2 - 23^2$

Obtain mentally the following quotients:

40. $(m^2 - n^2) \div (m - n)$
41. $(m^2 - 16) \div (m + 4)$
42. $(x^2 - 36) \div (x - 6)$
43. $(z^4 - 81) \div (z^2 + 9)$
44. $(4\,x^2 - y^2) \div (2\,x - y)$
45. $(16\,a^2 - 25\,b^4) \div (4\,a - 5\,b^2)$
46. $(49\,c^4 - 1) \div (7\,c^2 + 1)$
47. $(100\,m^4 - 49) \div (10\,m^2 - 7)$
48. $(169\,x^6 - 25) \div (13\,x^3 + 5)$
49. $(121\,x^2 - 256) \div (11\,x + 16)$

What binomial will divide each of the following? Give the resulting quotients.

50. $9\,x^2 - 16\,y^2$

51. $25\,r^4 - 81$

52. $36\,c^2 - 49\,d^6$

53. $100\,m^4 - 169$

54. $196\,a^2 - 256$

55. $400\,x^4 - \frac{1}{81}$

97. Complete factoring. When a number or an expression is to be factored, all its prime factors should be found. Often factors selected at first may be factored again.

Example 1. —

$$\begin{aligned} 48 &= 8 \cdot 6 \\ &= 4 \cdot 2 \cdot 2 \cdot 3 \\ &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3. \end{aligned}$$

Example 2. — Factor $36\,ty^8 - 36\,tz^8$ completely.

Solution. — 1. $36\,ty^8 - 36\,tz^8 = 36\,t(y^8 - z^8)$

2. $= 36\,t(y^4 + z^4)(y^4 - z^4)$

3. $= 36\,t(y^4 + z^4)(y^2 + z^2)(y^2 - z^2)$

4. $= 36\,t(y^4 + z^4)(y^2 + z^2)(y + z)(y - z).$

Rule. — To find the prime factors of an expression:

1. First remove any monomial factor which may be present.

2. Then factor the resulting expression, when possible, rewriting all expressions which cannot be factored.

NOTE 1. — Since only a few elementary cases in factoring are being taught in this chapter, you may at times get factors which are not factorable by any methods you know. "Factor completely," therefore, means *factor as far as possible*, using the methods taught in this chapter.

NOTE 2. — You probably think that $x^2 + y^2$ ought to be factorable. It is not. Neither is $x^4 + y^4$. You will learn more about this in Chapter XV.

EXERCISE 68

Factor as completely as you can the following:

1. $6\,m^2 - 6\,n^2$

2. $ac^2 - ad^2$

3. $4\,r^2 - 4\,s^4$

4. $5\,am^2 - 20\,an^2$

5. $16\,x^2y - 25\,yz^2$

6. $\frac{2}{3}\,x^2 - \frac{2}{3}\,y^4$

7. $75\,a^2 - 12\,b^2$

8. $81\,xt^2 - 49\,xy^2$

9. $32\,m^4 - 18\,n^2$

10. $\pi r^2 - \pi s^2$

11. $x^6z - y^4z$

12. $4\,x^4 - 4\,y^4$

13. $16\,at^4 - 16am^4$

14. $x^{12} - y^{12}$

15. $5\,x^8 - 5\,y^8$

CASE III

Type Form: $(ax + b)(cx + d)$

98. The product of two binomials of the form $(mx + p)$.

Development. — 1. Below, $(5x - 4y)$ is multiplied by $(2x + 3y)$.

$$\begin{array}{rrr} 5x & -\ 4y & \\ 2x & +\ 3y & \\ \hline 10x^2 & -\ 8xy & \\ & +\ 15xy & -\ 12y^2 \\ \hline 10x^2 & +\ 7xy & -\ 12y^2 \end{array}$$

$$\begin{array}{r} 5x - 4y \\ 2x + 3y \\ \hline -\ 8xy \\ +\ 15xy \\ \hline +\ 7xy \end{array}$$

Observe the products which give the middle term.

2. Similarly find the following products, and write the result thus:

$$(5x - 4y)(2x + 3y) = 10x^2 + 7xy - 12y^2$$

a. $(2x + 3)(3x + 1)$ *c.* $(4y - 3)(2y - 1)$
b. $(3m + 2)(2m + 3)$ *d.* $(3y + 2)(2y - 4)$

Notice that each of the products consists of three terms.

3. Examine carefully the results in Step 2. Then try to get the first and third terms of the following products mentally. Possibly you can also get the middle term mentally. Check by long multiplication.

a. $(2r + 4)(3r + 2)$ *c.* $(2s + 3)(2s + 1)$
b. $(3r + 5)(r + 2)$ *d.* $(4s + 2)(2s + 3)$

4. In all the examples of Step 3, the only difficulty is that of getting the "middle term." In Step 1, this middle term, $+7xy$, is the sum of $-8xy$ and $+15xy$. $-8xy$ is the product of $2x$ and $-4y$; $+15xy$ is the product of $5x$ and $3y$. These are called the **cross products**, because of the manner in which the arrows cross each other.

Observe the location of these cross products in the equation below.

1. $(5x - 4y)(2x + 3y) = 10x^2 + (-8xy + 15xy) - 12y^2$.
2. $\therefore (5x - 4y)(2x + 3y) = 10x^2 + 7xy - 12y^2$.

When finding such products mentally, write only the second line, guided by the

Rule. — 1. The first term of the product is the product of the first terms of the binomials. $[5x \cdot 2x = 10x^2]$

2. The middle term of the product is the algebraic sum of the cross products. [Follow the curved lines: $(-4y) \cdot (2x) = -8xy$, and $(5x) \cdot (+3y) = +15xy$; $(-8xy) + (+15xy) = +7xy$]

3. The third term of the product is the product of the second terms of the binomials. $[(-4y) \cdot (+3y) = -12y^2]$

Example. — Find $(5r - 6s)(2r + 3s)$.

Solution. — $(5r - 6s)(2r + 3s) = 10r^2 + 3rs - 18s^2$.

NOTE. — Only the line above should be written. The thinking, which produces it, is:

a. First term: $5r \cdot 2r = 10r^2$.

b. Second term: $-6s \cdot 2r = -12rs$; $5r \cdot 3s = +15rs$; $(-12rs) + (+15rs) = +3rs$.

c. Third term: $-6s \cdot +3s = -18s^2$.

EXERCISE 69

Find mentally:

1. $(2x + 1)(x + 2)$
2. $(2x + 1)(x + 3)$
3. $(3x + 1)(x + 2)$
4. $(3x + 1)(x + 3)$
5. $(3x + 1)(2x + 3)$
6. $(x - 4)(2x - 1)$
7. $(2x - 3)(x - 5)$
8. $(2x - 3)(2x - 5)$
9. $(3x - 2)(x - 6)$
10. $(2x - 3)(6x - 1)$
11. $(3a + 4)(2a - 1)$
12. $(4a + 5)(2a - 1)$
13. $(5a + 6)(3a - 2)$
14. $(6b - 5)(2b + 1)$
15. $(4m - 7)(2m + 3)$
16. $(5t - 8)(3t + 2)$
17. $(x - 7)(2x + 9)$
18. $(y + 11)(y + 6)$
19. $(z - 14)(z + 10)$
20. $(w + 20)(w - 5)$
21. $(3m + 2)(3m + 2)$
22. $(5m - 3)(5m - 3)$

23. $(6c + 5)(6c + 5)$
24. $(10k - 7)(10k - 7)$
25. $(12m + 5)(12m - 5)$
26. $(6ab - 5)(4ab + 1)$
27. $(7mn - 3)(5mn - 2)$
28. $(9r^2 - 2)(8r^2 + 3)$
29. $(13x^2 + 6)(5x^2 - 2)$
30. $(7p^2 - 8)(4p^2 + 5)$
31. $(10n^3 + 3)(12n^3 - 1)$
32. $(9c - 2d)(5c + 3d)$
33. $(11x - 5y)(2x + y)$
34. $(8a - 11b)(3a + 4b)$
35. $(7m + 9s)(3m - 5s)$
36. $(9t - 4x)(8t - 2x)$
37. $(2ab - 7c)(3ab + 5c)$
38. $(10mn^2 + 7)(4mn^2 - 3)$
39. $(5 + 3x)(2 + x)$
40. $(3y - 2z)(4y - 5z)$
41. $(9 + 4z^2)(10 - 5z^2)$
42. $(11 - 6xy)(5 + 6xy)$
43. $(12 - 5mn)(4 - 3mn)$
44. $(12a^2b - 2c)(a^2b - c)$
45. $(12x^2 + 11y)(2x^2 - y)$
46. $(8t^2 - 5x^2)(6t^2 - x^2)$
47. $(11m^2 - 4)(11m^2 + 4)$
48. $(x^2 - 6y)(x^2 + 13y)$
49. $(12a - 11c)(12a + 11c)$
50. $(15m - 4n)(6m + n)$
51. $(19a - 7b)(3a - b)$
52. $(9c^3 - 2d)(7c^3 + d)$
53. $(x^2 - 7y)(x^2 - 10y)$
54. $(11m^2 - 5n)(4m^2 + 5n)$
55. $(6p^2 - 7)(4p^2 + 5)$
56. $(13x^2 + 7z)(5x^2 - 2z)$
57. $(x - 8y)(\frac{1}{2}x + y)$
58. $(6a - \frac{1}{2}b)(6a + \frac{1}{3}b)$
59. $(\frac{1}{3}x - 8)(\frac{1}{2}x + 9)$
60. $(\frac{2}{3}a - \frac{1}{4})(12a - 15)$

99. Factoring trinomials of the form $mx^2 + nx + p$. The product of two binomials like $(2x + 3)$ and $(3x + 5)$ is a binomial of the form $mx^2 + nx + p$.

This means that there is a term containing the second power of x, one containing the first power (as a rule), and one free from x.

The following discussion shows how to factor trinomials of the form $mx^2 + nx + p$, when they are factorable.

Development. — 1. Find the products:
a. $(2x + 6)(3x + 2)$ *b.* $(3x - 5)(4x + 7)$
2. Factor $12x^2 + 23x + 5$.

Solution. — 1. The first terms of the binomials may be $2x$ and $6x$, for their product is $12x^2$. Place them in parentheses thus:

$$(2x \quad)(6x \quad).$$

The second terms must be 1 and 5, since these are the only factors of 5.

2. The second terms of the binomials are both positive since 5 and $23x$ are positive. Place the factors $+5$ and $+1$ in the parentheses and note the middle term which results.

a. $(2x+\mathbf{5})(6x+\mathbf{1})$; middle term, $+\mathbf{32}x$. Incorrect.

b. At once, interchange 5 and 1 in the parentheses. Thus:
$(2x+\mathbf{1})(6x+\mathbf{5})$; middle term, $+\mathbf{16}x$. Incorrect.

3. Step 2 shows that the factors $2x$ and $6x$ for $12x^2$ are incorrect. Try $3x$ and $4x$ for $12x^2$, thus: $(3x \quad)(4x \quad)$.

a. $(3x+\mathbf{1})(4x+\mathbf{5})$; middle term, $+\mathbf{19}x$. Incorrect.

b. $(3x+\mathbf{5})(4x+\mathbf{1})$; middle term, $+\mathbf{23}x$. Correct.

Check. — Does $(3x+5)(4x+1) = 12x^2+23x+5$? Yes.

NOTE. — This may seem a long process at first, but practice soon develops such skill that most of the trial of factors can be done mentally.

EXERCISE 70

Factor:

1. $2x^2+7x+3$

2. $6x^2+11x+3$

3. $6r^2+5r+1$

4. $2s^2+7s+5$

5. $3a^2+8a+5$

6. $4x^2+8x+3$

7. $6y^2+11y+3$

8. $2x^2+9x+10$

9. $3z^2+10z+8$

10. $12w^2+17w+6$

11. $5m^2-7m+2$

Solution. — 1. Since 2 is positive, its factors must have like signs and, since -7 is negative, the cross products must both be negative. Therefore the factors of 2 must both be negative.

(Complete the solution.)

12. $3m^2-7m+4$

13. $4p^2-9p+5$

14. $9s^2-9s+2$

15. $6t^2-11t+4$

16. $2w^2-11w+15$

17. $3w^2-17w+20$

18. $6y^2-19y+3$

19. $14y^2-11y+2$

20. $x^2-10xy+24y^2$

21. $9x^2-12xy+4y^2$

22. Factor $15x^2 + 14x - 8$.

Solution. — 1. Since -8 is negative, the factors of it must have unlike signs; since $14x$ is positive, the larger cross product must be positive.

2. For $15x^2$, try $(3x \qquad)(5x \qquad)$. For 8, try 2 and 4.

a. $(3x \qquad 2)(5x \qquad 4)$. The cross product, $3x \cdot 4$ is the larger; therefore make 4 positive and 2 negative.

Thus: $(3x - \mathbf{2})(5x + \mathbf{4})$. Middle term, $2x$. Incorrect.

b. At once, try $(3x \qquad \mathbf{4})(5x \qquad \mathbf{2})$. Now the cross product, $4 \cdot 5x$, is the larger. Make 4 positive and 2 negative.

Thus: $(3x + \mathbf{4})(5x - \mathbf{2})$. Middle term, $+14x$. Correct.

23. Factor $24m^2 - m - 10$.

Solution. — 1. The factors of -10 must have unlike signs. The signs must be selected so that the larger cross product is negative, since $-m$ is negative.

2. Try $(6m \qquad)(4m \qquad)$. For 10, try 5 and 2.

a. $(6m - \mathbf{5})(4m + \mathbf{2})$. Middle term, $-\mathbf{8}m$. Incorrect.

b. At once, try interchanging 5 and 2.

Thus: $(6m + \mathbf{2})(4m - \mathbf{5})$. Middle term, $-\mathbf{22}m$. Incorrect.

3. For $24m^2$, try $3m$ and $8m$. For 10, try 5 and 2.

a. Thus: $(3m - \mathbf{5})(8m + \mathbf{2})$. Middle term, $-\mathbf{34}m$. Incorrect.

b. Try $(3m - \mathbf{2})(8m + \mathbf{5})$. Middle term, $-m$. Correct.

NOTE. — In every case, *first select the proper factors to give the end terms; then decide which cross product is the larger; then give the second terms their signs so that this larger cross product has the same sign as the middle term.*

24. $2x^2 + 5x - 3$
25. $6a^2 - 5a - 6$
26. $3a^2 + 8a - 3$
27. $4b^2 - 5b - 6$
28. $5c^2 - c - 4$
29. $6d^2 + 7d - 5$
30. $5k^2 + 7k - 6$
31. $12m^2 - m - 6$
32. $12p^2 + 8p - 15$
33. $12x^2 - 13xy + 3y^2$
34. $18x^2 + 3xz - 10z^2$
35. $8t^2 - 26tw + 21w^2$
36. $16m^2 + 18mn - 9n^2$
37. $x^2 - 5xy - 24y^2$
38. $x^2 - wx - 110w^2$
39. $12r^2 - 13rs - 14s^2$
40. $9a^2 - 30ab + 25b^2$
41. $63 + 2x - x^2$
42. $14 - 29y - 15y^2$
43. $16 - 40x + 25x^2$

44. $8x^2 - 18xy + 7y^2$

45. $10z^4 - 21z^2 + 9$

46. $24c^2 - cd - 10d^2$

47. $28r^2 + 3rs - 18s^2$

48. $121 - 44x + 4x^2$

49. $x^2 + xy - 56y^2$

50. $18x^2 - 39x + 11$

51. $14r^2 - 61rt - 9t^2$

52. $63s^6 + s^3 - 20$

53. $40y^2 + yw - 6w^2$

54. $21 - 40x + 16x^2$

55. $27a^2 + 15ab - 50b^2$

56. $40x^2 - 18x - 9$

57. $36m^2 - 84mn + 49n^2$

58. $12r^2 - 43rt - 20t^2$

59. $20m^2 + 9mn - 20n^2$

CASE IV

Type Form: $(a + b)^2 = a^2 + 2ab + b^2$

TO THE TEACHER. — Since this type form is a special case of that taught in § 98 and § 99, the customary treatment of it is unnecessary, in large part. "Completing the square" appears again in the chapter on quadratics. However, *the following customary treatment of this type form* is included for those who have special reasons for teaching it. (*E.g.*, it appears in most recommended courses of study.)

If § 100 and § 101, which follow, are not taught, then Exercises 71 and 72 may be used for timed tests on the methods of § 98 and § 99 respectively. A conservative median speed is 20 examples in 10 minutes.

100. Squaring a binomial.

Development. — 1. $(a + b)^2 = (a + b)(a + b)$

$$= a^2 + 2ab + b^2. \qquad (\S\,98)$$

2. Similarly, $(a - b)^2 = (a - b)(a - b)$

$$= a^2 - 2ab + b^2.$$

These formulas are expressed by the rules which follow:

Rule 1. — To square the sum of two numbers:

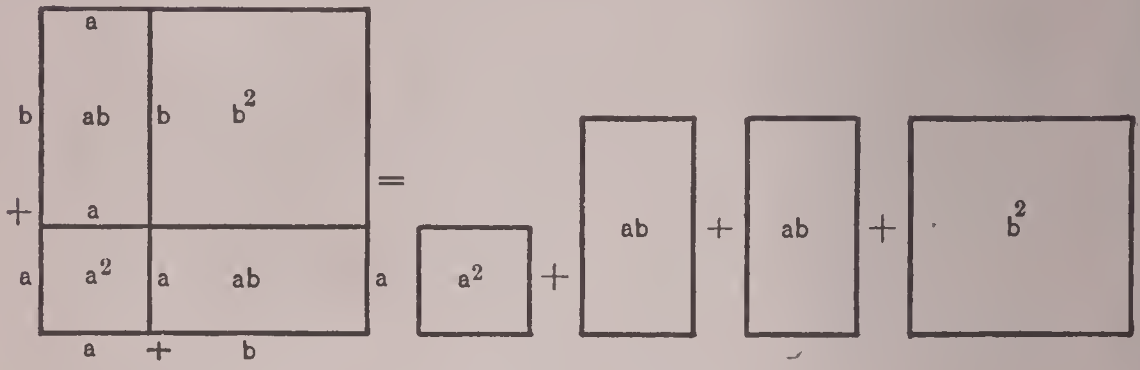
Square the first number; add twice the product of the first and second numbers; add the square of the second number.

Rule 2. — To square the difference of two numbers:

Square the first number; subtract twice the product of the two numbers; add the square of the second number.

Since a square whose side measures $(a + b)$ has for its area $(a + b)(a + b)$ square inches, the following figure illustrates $(a + b)^2$.

$$(a + b)^2 = a^2 + 2\,ab + b^2.$$



Example. — Find the square of $3\,x^2 - 4\,y^3$.

Solution. — You have two ways, now, of obtaining $(3\,x^2 - 4\,y^3)^2$.

a. Using Rule 2 above:

$$\begin{aligned}(3\,x^2 - 4\,y^3)^2 &= (3\,x^2)^2 - 2(3\,x^2)(4\,y^3) + (4\,y^3)^2\\ &= 9\,x^4 - 24\,x^2y^3 + 16\,y^6.\end{aligned}$$

b. Using the method of § 98:

$$\begin{aligned}(3\,x^2 - 4\,y^3)^2 &= (3\,x^2 - 4\,y^3)(3\,x^2 - 4\,y^3)\\ &= 9\,x^4 - 24\,x^2y^3 + 16\,y^6.\end{aligned}$$

EXERCISE 71

Square the following binomials by inspection:

1. $a + 7$
2. $b + 8$
3. $c - 5$
4. $d - 4$
5. $e + 9$
6. $x^2 + 10$
7. $r^2 - 11$
8. $s^2 - 12$
9. $ab + 20$
10. $2\,a + 3$
11. $4\,a - b$
12. $3\,c + d$
13. $4\,x - 3$
14. $5\,x + 2\,y$
15. $3\,x - 5\,y$
16. $2\,x^2 - 7$
17. $3\,x^2 + 6$
18. $5\,m + 8\,n$
19. $3\,xy - 7$
20. $4\,w^2 - 5\,z$
21. $6\,t + 9\,n$
22. $3 - 4\,x$
23. $5 - 7\,x$
24. $8\,y - 3\,x$
25. $3\,cd - 10$
26. $5\,r^2 - 9\,s$
27. $2\,x^2 - 11\,y^2$
28. $7\,c^3 - 4\,dm$
29. $11\,a^2 - 3\,b^3$
30. $12\,m^2 - 5\,n$
31. $m + \frac{1}{2}$
32. $n + \frac{1}{4}$
33. $x - \frac{1}{3}$

34. $y + \frac{2}{5}$

35. $z - \frac{2}{3}$

36. $w^2 + \frac{3}{4}$

37. $a - \frac{2}{7}$

38. $b + \frac{3}{8}$

39. $m - \frac{5}{6}$

40. Square 29 mentally.

Solution. — 1. $29^2 = (30 - 1)^2$
2. $= 900 - 60 + 1$, or 841.

Similarly square:

41. 19
42. 39
43. 49
44. 18
45. 28
46. 38
47. 21
48. 31
49. 22
50. 42
51. 51
52. 52

53. *Problem.* — Find a rule for squaring a number ending in 5.

Solution. — 1. $35 = 3 \cdot 10 + 5$; $45 = 4 \cdot 10 + 5$; $55 = 5 \cdot 10 + 5$.
2. Similarly, any number ending in 5 may be represented by $10\,n + 5$. Thus, for 95, n is 9, since $9 \times 10 + 5 = 95$.
3.
$$\begin{aligned}(10\,n + 5)^2 &= 100\,n^2 + 100\,n + 25\\ &= 100\,n(n + 1) + 25\end{aligned}$$
or, $n \cdot (n + 1)$ hundreds $+ 25$.
Thus, the square of 95, in which $n = 9$, is
$9 \cdot (9 + 1)$ hundreds $+ 25$, or 9025.

Rule. — To square a number ending in 5, drop the 5, multiply the balance of the number by the consecutive integer, and affix 25 to the result.

Example. — $85^2 = 7225$.

Explanation. — $8 \times 9 = 72$; affixing 25, the result is 7225.

54. Find by this rule the squares of some numbers ending in 5, such as 35, 45, 105, 115, etc.

101. **A perfect square trinomial** is the result obtained from squaring a binomial.

Thus, $(3\,x + 5\,y)^2 = 9\,x^2 + 30\,xy + 25\,y^2$.
Then $9\,x^2 + 30\,xy + 25\,y^2$ is a perfect square trinomial.
Similarly, $(3\,x - 5\,y)^2 = 9\,x^2 - 30\,xy + 25\,y^2$.
Then $9\,x^2 - 30\,xy + 25\,y^2$ is a perfect square trinomial.

Rule. — 1. A trinomial is a perfect square when two of its terms are perfect squares ($9x^2$ and $25y^2$) preceded by plus signs; and when the remaining term ($30xy$) is twice the product of the square roots of the perfect square terms, preceded by either a plus sign or a minus sign.

2. To find the square root of a perfect square trinomial: extract the square roots of the two perfect square terms, and connect them by the sign of the remaining term.

Example. — Is $4x^2 + 9y^4 - 12xy^2$ a perfect square?

Solution. — 1. $4x^2$ is a perfect square; its square root is $2x$.

2. $9y^4$ is a perfect square; its square root is $3y^2$.

3. $12xy^2 = 2(2x)(3y^2)$.

4. Hence, $4x^2 + 9y^4 - 12xy^2$ is a perfect square, and its square root is $2x - 3y^2$.

5. $\therefore\ 4x^4 + 9y^4 - 12xy^2 = (2x - 3y^2)^2$

or $= (2x - 3y^2)(2x - 3y^2)$.

EXERCISE 72

Supply the missing term so as to make perfect square trinomials of the following expressions, and then give the square roots.

1. $x^2 + (?) + 9$
2. $m^2 - (?) + 25$
3. $x^2 + (?) + 36a^2$
4. $y^2 - (?) + 4t^2$
5. $16x^2 + (?) + 1$
6. $9x^2 - (?) + 1$
7. $16x^2 - (?) + 9$
8. $4m^2 - (?) + 49n^2$
9. $25a^2 + (?) + 16b^2$
10. $64m^4 - (?) + 9c^2$

NOTE. — Further practice in "completing the square" is given in § 164, page 249.

Find the factors, if possible:

11. $c^2 - 12cd + 36d^2$
12. $x^4 + 14x^2y + 49y^2$
13. $m^2 - 24mn + 144n^2$
14. $25y^2 + 10yz - z^2$
15. $9x^2 - 30xy + 25y^2$
16. $16r^2 - 56rs^2 + 49s^4$

17. $9x^2 + 49y^2 - 42xy$
13. $4a^2 + 25x^2 - 20ax$
19. $81x^2 + 4y^2 - 36xy$
20. $9m^2 - 60mn^2 + 100n^4$
21. $36x^2 - 122xy + 12y^2$
22. $25c^4 - 30c^2d + 9d^2$
23. $49x^6 + 25y^2 - 70x^3y$
24. $64p^4 + 9q^2 - 48p^2q$
25. $9m^2 + 49n^2 + 63mn$
26. $81m^4 - 72m^2x + 16x^2$
27. $49x^2 + 36y^2 - 84xy$
28. $9t^2 - 60tw^2 + 100w^4$
29. $144p^2 - 84pw + 49w^2$
30. $121x^2 - 110wx + 25w^2$
31. $9x^2 + 15xy + 25y^2$
32. $16t^4 + 49w^2 - 56t^2w$
33. $49m^2 + 70mn - 25n^2$
34. $256r^2 - 96rs + 9s^2$

CASE V

Type Form: $(x + a)(x + b) = x^2 + (a + b)x + ab$

TO THE TEACHER. — Since this type form is a special case of that taught in § 98 and § 99, the customary treatment is unnecessary. However, *the customary treatment has been included,* for those who have special reasons for teaching it. (*E.g.*, it is specified in some courses of study.)

If § 102 and § 103 are not taught, Exercises 73 and 74 may be used as **timed tests on the methods of § 98 and § 99** respectively. A median speed of 20 examples in 10 minutes is a conservative standard.

102. The product of two binomials having a common term.

Development. — 1. Find by actual multiplication the following products, and write the results as in part *a*.

a. $(x + 2)(x + 3) = x^2 + 5x + 6$

b. $(x + 5)(x + 3) = ?$
c. $(a + 5)(a + 6) = ?$
d. $(m - 7)(m - 2) = ?$
e. $(s - 5)(s - 8) = ?$

2. Observe carefully the results in Step 1. Try to find the following products mentally; check the results by multiplication.

a. $(b + 4)(b + 2)$
b. $(c + 4)(c + 3)$
c. $(x - 3)(x - 7)$
d. $(y - 4)(y - 5)$
e. $(r + 6)(r + 10)$
f. $(x - 5)(x - 11)$

Rule. — To obtain the product of two binomials having a common term:

1. Square the common term.
2. Multiply the common term by the algebraic sum of the second terms of the binomials.
3. Find the product of the second terms.
4. Add the results.

Example. — Find $(ab + 2)(ab - 11)$.

Solution. — $(ab + 2)(ab - 11) = a^2b^2 - 9\,ab - 22$.

NOTE. — Here, glance at $+2$ and -11; note that their sum is -9, and that their product is -22; write the result as above.

EXERCISE 73

Find the following products:

1. $(x + 3)(x + 6)$
2. $(x + 2)(x + 5)$
3. $(x + 2)(x + 7)$
4. $(a + 4)(a + 3)$
5. $(a + 5)(a + 6)$
6. $(b - 3)(b - 5)$
7. $(c - 7)(c - 2)$
8. $(d - 8)(d - 5)$
9. $(e - 9)(e - 4)$
10. $(x - 11)(x - 5)$
11. $(y + 6)(y - 2)$
12. $(z + 10)(z - 3)$
13. $(w + 11)(w - 5)$
14. $(w - 12)(w + 3)$
15. $(x - 13)(x + 10)$
16. $(y - 11)(y + 6)$
17. $(z + 10)(z - 8)$
18. $(k + 15)(k - 8)$
19. $(m + 20)(m - 4)$
20. $(r - 16)(r + 5)$
21. $(s^2 - 7)(s^2 - 6)$
22. $(t^3 - 9)(t^3 + 7)$
23. $(a - 8\,b)(a - 4\,b)$
24. $(c + 5\,d)(c + 9\,d)$
25. $(x + 9\,y)(x - 2\,y)$
26. $(r - 12\,s)(r + 2\,s)$
27. $(xy + 5)(xy - 14)$
28. $(m^2 - 6\,n)(m^2 + 13\,n)$
29. $(c^3 - 4\,d)(c^3 - 15\,d)$
30. $(x^2 + 7\,y^2)(x^2 - 6\,y^2)$
31. $(a - 11\,b)(a + 12\,b)$
32. $(r^2 - 9\,s)(r^2 + 20\,s)$

33. $(rs - 14\,t)(rs + 2\,t)$
34. $(t - 16\,w)(t + 6\,w)$
35. $(x^2 - 17)(x^2 - 5)$
36. $(y^2 - 2)(y^2 + 19)$
37. $(z - 3\,w)(z + 21\,w)$
38. $(m^2 - 13\,x)(m^2 + 8\,x)$
39. $(t - 11\,w)(t + 10\,w)$
40. $(x^2 - 16)(x^2 + 15)$
41. $(m - 25)(m + 20)$
42. $(a - 19\,b)(a + 4\,b)$
43. $(x + 26)(x - 5)$
44. $(c - 30)(c + 10)$
45. $(m + 20)(m - 15)$
46. $(x + \frac{1}{2})(x + \frac{1}{4})$
47. $(x - \frac{1}{3})(x - \frac{2}{3})$
48. $(r - \frac{1}{2})(r - 2)$
49. $(s + \frac{1}{2})(s - 1)$
50. $(t - \frac{1}{3})(t + 1)$

51. Find the product 62×68.

Solution. — 1. $62 \times 68 = (60 + 2)(60 + 8)$
2. $= 60^2 + (10 \cdot 60) + 16.$
3. $\therefore\ 62 \times 68 = 3600 + 600 + 16$, or 4216.

NOTE. — Learn to do all this solution mentally.

Find the following products, mentally, if you can:

52. $23 \cdot 27$
53. $32 \cdot 38$
54. $24 \cdot 26$
55. $43 \cdot 47$
56. $52 \cdot 58$
57. $41 \cdot 49$
58. $22 \cdot 26$
59. $32 \cdot 34$
60. $43 \cdot 45$
61. $33 \cdot 36$
62. $23 \cdot 25$
63. $41 \cdot 45$

103. Factoring trinomials of the form $x^2 + px + q$.

Development. — 1. $(x + 5)(x - 3) = x^2 + 2\,x - 15.$

In obtaining this product, the algebraic sum of $+5$ and -3 is taken for the coefficient of x, and the product of $+5$ and -3 is taken for the third term.

To factor $x^2 + 2\,x - 15$, then, it is necessary to find two numbers whose product is -15, and whose algebraic sum is $+2$.

2. Factor $x^2 + 7\,x + 12$.

Solution. — 1. Two numbers whose product is $+12$, and whose sum is $+7$, are $+3$ and $+4$.

2. The factors are $(x + 3)(x + 4)$

Check. — Does $(x + 3)(x + 4) = x^2 + 7\,x + 12$? Yes.

Rule. — To factor a trinomial of the form $x^2 + px + q$:

1. Find two numbers whose algebraic product is q and whose algebraic sum is p.

2. One factor is x + one number; the other factor is x + the other number.

EXERCISE 74

Factor the following trinomials:

1. $x^2 + 8x + 12$
2. $x^2 + 10x + 24$
3. $r^2 + 11r + 24$
4. $s^2 + 11s + 30$
5. $t^2 + 10t + 16$
6. $y^2 + 12y + 27$
7. $z^2 + 14z + 40$
8. $m^2 + 14m + 33$
9. $p^2 - 7p + 12$

Hint. — Since 12 is positive, the second terms must have like signs; and, since the middle term is negative, these second terms, having like signs, must be negative.

(Complete the solution.)

10. $x^2 - 7x + 10$
11. $y^2 - 10y + 21$
12. $z^2 - 9z + 20$
13. $a^2 - 12a + 35$
14. $b^2 - 9b + 18$
15. $m^2 - 9m + 14$
16. $r^2 - 12r + 32$
17. $s^2 - 10s + 9$
18. $x^2 - 26x - 192$

Solution. — 1. Since -192 is negative, the second terms must have unlike signs. Since -26 is negative, and is the sum of the second terms, the term having the larger absolute value must be the negative one.

2. Since 192 is a large number, systematically list its factors.

$(+1)(-192)$; $(+1) + (-192) = -191$. Unsatisfactory.
$(+2)(-96)$; $(+2) + (-96) = -94$. Unsatisfactory.
$(+3)(-64)$; $(+3) + (-64) = -61$. Unsatisfactory.
$(+4)(-48)$; $(+4) + (-48) = -44$. Unsatisfactory.
$(+6)(-32)$; $(+6) + (-32) = -26$. These are satisfactory.

3. $\therefore\ x^2 - 26x - 192 = (x + 6)(x - 32).$

Note. — Step 2 should be done mentally.

19. $x^2 + 4x - 12$

20. $y^2 + 2y - 15$

21. $z^2 + 5z - 14$

22. $w^2 - 3w - 18$

23. $t^2 - 3t - 28$

24. $m^2 - 4m - 45$

25. $r^2 - r - 42$

26. $s^2 + 6s - 27$

27. $t^2 - 5t - 50$

28. $z^2 + 9z - 36$

29. $a^2 - 6a - 55$

30. $a^2 + 7ab - 18b^2$

31. $c^2 - 5cd - 24d^2$

32. $w^2 + 6wx - 40x^2$

33. $p^4 + 4p^2 - 77$

34. $r^2 - rs - 90s^2$

35. $m^4 - 10m^2p - 24p^2$

36. $x^4 - 2x^2y - 63y^2$

37. $t^2 + 2tw - 48w^2$

38. $a^4 - a^2b - 20b^2$

39. $c^6 + c^3 - 56$

40. $m^6 + 12m^3 - 13$

41. $w^4 - 4w^2 - 32$

42. $x^2 + 11xy - 26y^2$

43. $r^2 - 2rs^2 - 24s^4$

44. $a^4 + 4a^2c - 21c^2$

45. $b^4 + 3b^2 - 40$

46. $x^4 - x^2y^2 - 12y^4$

47. $y^4 - y^2w - 30w^2$

48. $w^2 - wz - 72z^2$

104. Mental algebra. The purpose of the study of special products and factoring is to enable you to do much of the algebraic computation mentally. **Avoid the use of "scratch" paper.** Do the multiplying and factoring mentally, and then write the results.

Example. — Find the simplest form of

$$3a(2a + 7)(a - 3) - 4a(a + 5)(a - 6).$$

Solution. — 1. $3a(2a + 7)(a - 3) - 4a(a + 5)(a - 6)$

2. $= 3a(2a^2 + a - 21) - 4a(a^2 - a - 30)$

3. $= 6a^3 + 3a^2 - 63a - 4a^3 + 4a^2 + 120a$

4. $= 2a^3 + 7a^2 + 57a.$

NOTE 1. — The equality sign in line 2 means that the expression in line 1 equals that in line 2, etc.

NOTE 2. — The binomials are multiplied together first (Step 2), and then their products are multiplied by the monomials in Step 3. Thus, $(2a + 7)(a - 3)$ gives $2a^2 + a - 21$; and $-4a(a^2 - a - 30)$ gives $-4a^3 + 4a^2 + 120a$.

EXERCISE 75

Find (without use of any scratch paper):

1. $2m(4m - 1)(m + 2)$
2. $-3(5 - x)(6 + x)$
3. $+4(2c - 3d)(2c - 3d)$
4. $-5(x - 4y)(x + 4y)$
5. $+6(3a - 2b)(a + 3b)$
6. $-2(2r + 5s)(2r - 5s)$
7. $+8(m - 11n)(m + 4n)$
8. $-a(8a - 7m)(2a + 3m)$

Find all the prime factors of the following. (Review § 97.)

9. $3s^3 - 13s^2 - 10s$
10. $18at^2 + 48at + 32a$
11. $20m^2r - 125n^2r$
12. $18a^2c^2 - 57ac^2 + 45c^2$
13. $9x^2n^3 - 16n^3$
14. $42t - 15tx - 3tx^2$
15. $6r^4 + 7r^3 - 5r^2$
16. $6mx^2 - 13mxy + 6my^2$
17. $6x^2 + 23x^3 - 18x^4$
18. $4x^2y - 12xy^2 + 9y^3$

Solve the following equations:

19. $2(3x - 1)(x + 2) - 3(x - 2)(2x + 1) = 40$

Solution. — 1. $2(3x - 1)(x + 2) - 3(x - 2)(2x + 1) = 40.$
2. $\therefore 2(3x^2 + 5x - 2) - 3(2x^2 - 3x - 2) = 40.$
3. $\therefore \cancel{6}\,\cancel{x^2} + 10x - 4 - \cancel{6}\,\cancel{x^2} + 9x + 6 = 40.$

(Complete the solution.)

20. $(4y - 3)(y + 2) - 4(y - 3)(y + 3) = 35$
21. $(2 - 3t)(2 - 3t) - (3t - 1)(3t + 1) = 2$
22. $5(2r - 1)(3r + 1) - (5r + 2)(6r - 3) = 2$
23. $3(2a - 5)(a + 4) - 6(a + 8)(a - 2) = 9$
24. $7(a - 1)(2a + 1) - (7a - 1)(2a + 1) = -2$
25. $(5x - 4)(5x - 4) - 5(5x + 2)(x - 4) = 6$

26. A certain number exceeds another number by 5. The square of the larger exceeds the square of the smaller by 95. What are the numbers?

27. Separate 25 into two parts such that the square of the larger exceeds the square of the smaller by 225.

28. A certain number is 8 more than another number. The product of the two numbers exceeds the square of the smaller by 40. What are the two numbers?

29. There are three consecutive integers such that the product of the second and third exceeds the square of the first by 47. What are the integers?

30. There are three consecutive integers such that the square of the second, diminished by the square of the first, gives a remainder which is 11 less than the third. What are the integers?

31. Find three consecutive integers such that twice the square of the third exceeds the sum of the squares of the first and second by 43.

EXERCISE 76

Supplementary Miscellaneous Examples

Multiply mentally:

1. $(4x + 5y)(4x - 5y)$

2. $(\frac{2}{3}a - b)(\frac{2}{3}a + b)$

3. $(a^2 - 3b)(a^2 - 3b)$

4. $(5r^2 - 7s)^2$

5. $(6a^2 + b)(3a^2 - 7b)$

6. $(x + 2)(x - 2)(x^2 + 4)$

7. $(11m - 5n)(m + n)$

8. $3(a - x)(a - 2x)$

9. $(4a - 3b^2)(3a + 4b^2)$

10. $-2m(m^2 - mn + n^2)$

11. $6p(x^2 - 1)(x^2 + 1)$

12. $-4(2x + 1)(3x - 5)$

13. $(a^2 - 11c)(a^2 + 5c)$

14. $(6x^2 - y)^2$

15. $(x + \frac{1}{2})(x + \frac{1}{3})$

16. $(x - \frac{3}{4})(x - \frac{1}{4})$

17. $(y + \frac{1}{2})(y - \frac{1}{4})$

18. $(\frac{1}{2}a + 1)(\frac{1}{2}a - 2)$

19. $3x(x - 2)(2x + 3)$

20. $(B^2 - AC)(B^2 + 5AC)$

21. $2a(a^2 - 3)(a^2 + 3)$

22. $-4(x + 2y)(x - 2y)$

23. $(\frac{3}{5}t - \frac{1}{4})(\frac{3}{5}t + \frac{1}{4})$

24. $91 \cdot 89$

25. $72 \cdot 78$

26. $68 \cdot 72$

27. $(x - 1)(x + 1)(x^2 + 2)$

28. $(a + 3)(a - 3)(a^2 + 5)$

29. $(3n - 4m)^2$

30. $(6p^2 - 5q)^2$

Find the prime factors of the following. (Review § 97.)

31. $ax^2 - 2\,ax - 35\,a$

32. $3\,c^2 + 4\,c - 7$

33. $6\,s^2t - 7\,st - 5\,t$

34. $16\,a^4 - 80\,a^2 + 100$

35. $3\,x^2 + 9\,x - 120$

36. $5\,r^2 + 16\,r + 3$

37. $18\,x^2y^2 - 3\,xy^2 - 10\,y^2$

38. $3\,m^2 - 30\,m + 75$

39. $2\,n^2 - 14\,n - 36$

40. $18\,x^2 + 17\,x - 15$

41. $15\,t^2 - 6\,t - 21$

42. $4\,r^2s^2 + 4\,rs^2 - 15\,s^2$

43. $\frac{1}{25}\,a^4 - \frac{9}{49}\,c^2$

44. $5\,pr^2 + 5\,pr - 10\,p$

45. $3\,z^2 + 7 + 22\,z$

46. $21\,a^2 - 10\,b^2 - 29\,ab$

47. $m^2x^2 - 7\,mx - 44$

48. $2\,aw^2 - 3\,aw - 20\,a$

49. $169\,m^2 - 26\,mn + n^2$

50. $w^6 + 72 - 18\,w^3$

51. $8\,a^4 + 18\,a^2 - 35$

52. $4\,c^2d - 8\,cd - 21\,d$

53. $4\,a^2y^2 - 28\,ay^2 + 49\,y^2$

54. $z^2 + 2\,zt - 63\,t^2$

55. $20\,c^2 - 30\,cd - 140\,d^2$

56. $10\,x^2 - 13\,xy - 30\,y^2$

57. $9\,a - am^2$

58. $a^2 - 36\,b^2 + 5\,ab$

59. $9\,x^2 + 3\,x - 56$

60. $10\,a^2 + 19\,a - 15$

Solve the following equations. (See Example 19, page 142.)

61. $(3\,x - 5)(2\,x + 4) = (3\,x - 4)(2\,x + 7) - 3$

62. $(6\,a + 1)(3\,a - 4) - 2(3\,a - 2)^2 = 3$

63. $(2\,m - 5)^2 = (4\,m - 3)(m + 1)$

64. $(5 - 3\,s)(2 + s) - 3(1 - s)(1 + s) = 10$

65. $2(5\,t - 2)(3\,t + 1) - 3(2\,t + 3)(5\,t - 2) = 49$

66. $3\,y(y + 4) - (y + 1)(2\,y - 1) = (y + 1)(y + 9)$

67. $5(2\,a - 1)^2 - (4\,a - 3)(4\,a + 3) = 2(2\,a + 7)(a - 5)$

68. $(1 - 3\,x)^2 - (2\,x - 1)^2 = 5(x + 1)(x - 7) - 21$

69. $(4\,m + 1)(3\,m - 2) - 2(2\,m - 1)(m + 5) = (8\,m - 3)(m - 2)$

70. $(7\,z - 3)(7\,z + 3) - 8(5\,z + 1)(z - 2) = (3\,z + 11)^2$

EXERCISE 77

Problems about Areas

1. *a.* What is the area of the rectangle whose base is $(x - 2)$ inches and altitude is $(x + 5)$ inches?

b. What is the perimeter of this rectangle?

2. The base of a certain rectangle exceeds its altitude by 5 inches.

a. If the altitude is h in., what is the length of the base?

b. What is the area of the rectangle?

c. What is the area of the square whose side equals the base of this rectangle?

d. Write the equation which expresses the fact that the area of the square is 65 sq. in. more than the area of this rectangle.

e. Solve this equation and find the dimensions of the rectangle and of the square.

NOTE. — The essential facts of this example can be charted thus:

	BASE	ALTITUDE	AREA
Rectangle	$h + 5$	h	$h(h + 5)$
Square	$h + 5$	$h + 5$	$(h + 5)^2$

$$\therefore\ (h + 5)^2 = h(h + 5) + 65.$$

3. *a.* Express the area of the square whose side is s inches.

b. Express the base, altitude, and area of the rectangle whose base is 10 inches longer and whose altitude is 2 inches longer than the side of the square.

c. Arrange the results of parts *a* and *b* in a chart as in the Note above.

d. Write the equation which expresses the fact that the area of the rectangle is 200 sq. in. more than the area of the given square.

e. Solve the equation, and find the dimensions of the square and the rectangle. Check the solution.

4. The base of a certain rectangle exceeds its altitude by 9 inches. A better shaped rectangle is formed by making the altitude 3 in. longer and the base 2 in. longer; but this increases the area by 78 sq. in. What are the dimensions of the two rectangles? (Represent the dimensions; form an equation, and solve it.)

5. The base of a certain rectangle exceeds its altitude by 8 inches. If the base and altitude are both decreased by 4 inches, the old area exceeds the new area by 192 square inches. What are the dimensions of the rectangle?

6. The base of a certain rectangle is 7 feet more than its altitude. If the base be increased 5 feet and the altitude be decreased 3 feet, the area will remain unchanged. What are the dimensions of the rectangle?

7. The base of a certain rectangle exceeds twice its altitude by 3 inches. The area of a square whose side equals the base exceeds twice the area of the rectangle by 57 square inches. What are the dimensions of the rectangle?

8. A man had plans for a house whose foundation was 12 feet longer than it was wide. Finding that it was too expensive, he cut off 3 feet from the length and 2 feet from the width. The new foundation covered 158 square feet less than the old. What were the dimensions before and after making the change?

9. The main shaft of Washington's Monument is square at the bottom and top. The side of the lower square exceeds the side of the upper square by 21 feet. The area of the lower square exceeds the area of the upper square by 1869 square feet. Find the dimensions of the two squares.

10. The base of a certain rectangle exceeds its altitude by 7 inches. A second rectangle is formed by making the altitude 3 inches shorter and the base 10 inches longer. The new rectangle contains 75 square inches more than the first rectangle. What are the dimensions of the first rectangle?

11. A man's vegetable garden is 100 feet longer than it is wide. A garden magazine article recommended that a 10-foot strip be reserved around all four sides as a drive for a team and wagon. He found that this would decrease the area of his lot available for gardening, by 9600 square feet. What are the dimensions of his garden?

105. Quadratic equations solved by factoring.

A new kind of equation can now be solved.

Example. — Find the number whose square exceeds the number itself by 6.

Solution. — 1. Let $n =$ the number.

2. Then $n^2 =$ the square of the number.

3. Then $n^2 = n + 6$.

4. $\therefore n^2 - n - 6 = 0$.

5. Factoring, $(n - 3)(n + 2) = 0$.

6. The question is, what value, or values, of n makes the left side have the value 0?

If n has the value which makes $n - 3 = 0$, then at once $(n - 3)(n + 2) = 0$, for $0 \cdot (n + 2) = 0$. This value of n is $+3$, for $3 - 3 = 0$.

Also, if n has the value which makes $n + 2 = 0$, then again $(n - 3)(n + 2) = 0$, for $(n - 3) \cdot 0 = 0$. This value of n is -2, for $-2 + 2 = 0$.

It seems, then, that $+3$ and -2 are both roots of the equation.

Check. — Does $3^2 - 3 - 6 = 0$? Does $9 - 9 = 0$? Yes.

Does $(-2)^2 - (-2) - 6 = 0$? Does $4 + 2 - 6 = 0$? Yes.

Also, both of these numbers satisfy the conditions of the problem.

That is, $3^2 = 9$; 9 exceeds 3 by 6.

$(-2)^2 = 4$; 4 exceeds -2 by 6.

Therefore "the number" is either $+3$ or -2.

NOTE. — If we were seeking *a number* which satisfied the equation 4, or the conditions of the given problem, then *either* 3 *or* -2 would do; but we are really seeking *all the numbers* satisfying equation 4 or the conditions of the problem; then *both* 3 and -2 must be accepted as proper results.

106. An equation like $n^2 - n - 6 = 0$ is called a **quadratic equation or an equation of the second degree.**

Other examples are: $4x^2 - 9 = 0$

and $3y^2 - \frac{2y}{5} = \frac{5}{3}y + 9.$

Notice that the equation has only one unknown; that this unknown does not appear in the denominator of any fraction; that it does appear with exponent 2; that it may or may not appear with exponent 1.

Every quadratic equation has two roots, just as the equation in § 105.

107. Solution of equations by factoring depends upon the following fact:

If one of the factors of a product is zero, the product also is zero.

Thus, $3 \times 0 = 0$; $(-5) \times 0 = 0$; $2 \times 0 \times (-3) = 0$.

Moreover, if a product of two or more factors is zero, then one, or more, of the factors must be zero.

Thus, if $(x - 1)(x + 2) = 0$, then $x - 1$ must $= 0$, *or* $x + 2$ must $= 0$, *or else both* $x - 1$ and $x + 2$ must $= 0$.

Rule. — To solve an equation by factoring:

1. **Transpose all terms to the left member.**
2. **Factor the left member completely.**
3. **Set each factor equal to zero, and solve the resulting equations.**
4. **The roots obtained in Step 3 are the roots of the given equation.**
5. **Check by substitution in the given equation.**

Example. — Solve the equation $\frac{m^2}{3} - \frac{m}{2} = \frac{35}{6}$.

Solution. — 1. M_6 $2m^2 - 3m = 35.$
2. S_{35} $2m^2 - 3m - 35 = 0.$
3. Factor, $(2m + 7)(m - 5) = 0.$

4. $2m + 7 = 0$ if $m = -\frac{7}{2}$.
$m - 5 = 0$ if $m = +5$.
5. The roots are $+5$ and $-\frac{7}{2}$.

NOTE. — Step 4 should be done mentally and omitted.

EXERCISE 78

Solve by factoring the following quadratic equations:

1. $x^2 - 10x + 21 = 0$
2. $x^2 + 7x + 10 = 0$
3. $y^2 - 5y - 24 = 0$
4. $t^2 = 9 - 8t$
5. $s^2 + 7s = 44$
6. $x^2 - x = 42$
7. $y^2 = 18y - 80$
8. $w^2 - 36 = 5w$
9. $x^2 - 25 = 0$
10. $4m^2 - 49 = 0$
11. $4c^2 + 5c - 6 = 0$
12. $5m^2 - 6 = 13m$
13. $2a^2 + 7a - 4 = 0$
14. $12s^2 - 17s + 6 = 0$
15. $15t^2 + 22t + 8 = 0$
16. $x^2 - 9x = 0$
17. $2y^2 + 7y = 0$
18. $5m^2 + 27m = 18$
19. $x^2 - \frac{15}{4}x - 1 = 0$
20. $x^2 + \frac{17}{3}x = 2$

108. A literal equation is one in which some or all of the known numbers are represented by letters; as,

$$2x + a = bx^2 - 10.$$

Example. — Find two numbers whose difference is a, and whose product is 6 times the square of a.

Solution. — 1. Let $x =$ the larger number.
2. Then $x - a =$ the smaller number.
3. $\therefore x(x - a) = 6a^2$.
4. $\therefore x^2 - ax - 6a^2 = 0$.
5. $\therefore (x - 3a)(x + 2a) = 0$.
6. $\therefore x = 3a$, or $x = -2a$, the larger number.
7. When $x = 3a$, then $x - a = 2a$, the smaller number.

Check. — Does $3a \times 2a = 6a^2$? Yes.

8. When $x = -2a$, then $x - a = -3a$, the smaller number.

Check. — Does $(-2a)(-3a) = 6a^2$? Yes.

(Continued on page 150.)

Therefore there are two sets of two numbers which satisfy the conditions of the problem: one set, $3a$ and $2a$; the other set, $-2a$ and $-3a$.

These are really formulæ for definite numbers satisfying the conditions of the problem.

Thus, if a is 7, one set is 21 and 14; the other set is -14 and -21. (Read the problem, supplying 7 for a, and then check these numbers 21 and 14; also -21 and -14.)

The number a can have any numerical value.

Example 2. — Solve the equation $3c^2 - 2c = 0$.

Solution. — 1. $3c^2 - 2c = 0.$

2. Factoring, $c(3c - 2) = 0.$

3. $\therefore\ c = 0$, and $c = \frac{2}{3}$.

NOTE. — In this example, the first factor is c. Following the third direction of the rule, c must $= 0$.

Check. — Does $3 \cdot 0^2 - 2 \cdot 0 = 0$? Does $0 - 0 = 0$? Yes.

Does $3(\frac{2}{3})^2 - 2(\frac{2}{3}) = 0$? Does $\cancel{3} \cdot \frac{4}{\underset{3}{\cancel{9}}} - \frac{4}{3} = 0$? Yes.

EXERCISE 79

1. $x^2 - 7ax + 12a^2 = 0$

2. $x^2 + 7bx + 10b^2 = 0$

3. $x^2 - 5mx - 6m^2 = 0$

4. $x^2 + 5tx - 24t^2 = 0$

5. $3x^2 - 7cx + 2c^2 = 0$

6. $2x^2 = 5bx + 3b^2$

7. $x^2 + 5nx = 66n^2$

8. $3x^2 + 13sx - 10s^2 = 0$

9. $x^2 - 6ax = 0$

10. $15x^2 = 2p^2 - 7px$

11. $9x^2 - 4a^2 = 0$

12. $5x^2 - 3cx = 0$

13. $2x^2 + \frac{3xw}{7} - \frac{2w^2}{7} = 0$

14. $x^2 = \frac{1}{8}t^2 - \frac{tx}{4}$

15. $\frac{x^2}{2} + \frac{3rx}{10} - \frac{r^2}{5} = 0$

16. $3x^2 = \frac{5ax}{8} + \frac{a^2}{8}$

17. $\frac{7x^2}{6} - \frac{dx}{2} = \frac{2d^2}{3}$

18. $\frac{x^2}{6} = \frac{17rx}{12} + \frac{3r^2}{4}$

19. $\frac{2x^2}{5} + \frac{sx}{2} - \frac{21s^2}{10} = 0$

20. $2x^2 + \frac{9mx}{20} = \frac{m^2}{2}$

109. Problems solved by quadratic equations.

Example 1. — Find three consecutive even integers such that twice the square of the second, increased by the product of the first and the third, is 296.

Solution. — 1. Let $x =$ the first of these three integers.
$\therefore x + 2 =$ the second of these three integers.
and $x + 4 =$ the third of these three integers.
2. $\therefore 2(x + 2)^2 + x(x + 4) = 296.$
3. $\therefore 2x^2 + 8x + 8 + x^2 + 4x - 296 = 0.$
4. $\therefore 3x^2 + 12x - 288 = 0.$
5. D_3 $x^2 + 4x - 96 = 0.$
6. Factoring, $(x + 12)(x - 8) = 0.$
7. $\therefore x = -12$, or $x = 8$, the first of the three integers.
8. When the first integer is -12, the next two even integers are -10 and -8.

Check. — Does $2(-10)^2 + (-12)(-8) = 296$?
Does $200 + 96 = 296$? Yes.

9. When the first integer is 8, the next two even integers are 10 and 12.

Check. — Does $2(10)^2 + 8 \times 12 = 296$?
Does $200 + 96 = 296$? Yes.

In this example, two sets of even integers are obtained, which satisfy the conditions of the problem.

Example 2. — Determine the base and altitude of the rectangle whose area is 140 square inches, and whose base exceeds its altitude by 13 inches.

Solution. — 1. Let $a =$ the number of inches in the altitude.
2. $\therefore a + 13 =$ the number of inches in the base.
3. $\therefore a(a + 13) =$ the area of the rectangle.
4. $\therefore a(a + 13) = 140.$
5. $\therefore a^2 + 13a = 140$, or $a^2 + 13a - 140 = 0.$
6. $\therefore (a + 20)(a - 7) = 0.$
7. $\therefore a = -20$, or $a = 7$, the number of inches in the altitude.
8. -20 is impossible as altitude of a rectangle.
9. When $a = 7$ in., then $a + 13$, or 20 in. $=$ the base.

Check. — Does $7 \times 20 = 140$? Yes.

In this example, while the quadratic has two roots, only one can have real meaning for the given problem.

EXERCISE 80

1. The square of a certain number equals the sum of 5 times the given number and 6. What is the number?

2. The square of a certain number diminished by 8 times the number itself is 33. What is the number?

3. If the square of a certain number be increased by 9 times the number and 25, the result is 7. What is the number?

4. The square of a certain number increased by the square of the number which is 2 more than the first gives the result 100. What are the numbers?

5. John is 4 times as old as Charles. If John's age be added to the square of Charles' age, the sum is 45. What are their ages?

6. The number of members in Edward's family is 2 more than the number in the family of his cousin, Fred. The square of the number in Fred's family is one less than twice the number in Edward's family. How many are there in each family?

7. The sum of the squares of certain two consecutive integers is 145. What are the integers?

8. There are three consecutive integers such that the square of the first, increased by the square of the second, and this result, diminished by the square of the third, gives 96. What are the integers?

9. There are three consecutive *odd* integers such that the sum of the squares of the second and third, increased by twice the square of the first, equals 300. What are the integers?

10. Separate 22 into two parts such that the square of the larger increased by the square of the smaller is 260.

11. Twice the square of a certain number equals the number itself. What is the number?

12. 9 times the square of a certain number is 25. What is the number?

13. The square of the number of years in the age of Mary, increased by 4, equals 200. How old is Mary?

14. The number of boys in a certain algebra class was 3 less than the number of girls. One fourth of the square of the number of girls exceeded 4 times the number of boys by 5. How many girls and how many boys were there in the class?

15. The area of a certain small piece of ground is 240 square rods. Its length is 8 rods more than its width. What are its dimensions?

16. How long and wide must a lot be to contain 4500 square feet, and have the length 5 times the width?

17. What are the dimensions of a triangle whose area is 54 square inches if the base is 3 times as long as the altitude?

18. What must be the dimensions of a trapezoid whose area is 96 square inches, if the upper base and the altitude are to be equal, and the lower base is to be twice the upper base?

19. A certain lot is 25 feet wide and 130 feet long. The owner desires to add 1400 square feet to the lot by increasing its width by some amount, and its length by 5 times the same amount. What must that amount be?

20. What must be the dimensions of a lot containing 8000 square feet, such that the length is 10 feet more than 3 times the width?

21. There are three consecutive integers such that the square of the first, increased by the square of the second, is 115 less than twice the square of the third. What are the integers?

22. A certain lot is 30 feet wide and 140 feet long. By increasing the width by a certain number of feet, and decreasing the length by twice the same number of feet, the area is increased 350 square feet. What is the number of feet?

SUPPLEMENTARY TOPIC

110. **The highest common factor** (H. C. F.) of two or more expressions is the product of all their common prime factors.

Example 1. — Find the H. C. F. of $12x^2$, $18x^3y^2$, and $24x^5y$.

Solution. — 1. Factor each expression.

a. $12x^2 = 2 \cdot 2 \cdot 3 \cdot x^2 = 2^2 \cdot 3 \cdot x^2.$

b. $18x^3y^2 = 2 \cdot 3 \cdot 3 \cdot x^3y^2 = 2 \cdot 3^2 \cdot x^3y^2.$

c. $24x^5y = 2 \cdot 2 \cdot 2 \cdot 3 \cdot x^5y = 2^3 \cdot 3 \cdot x^5y.$

2. *a.* 2 is the highest power of 2 which is a common factor, *because only one* 2 *is a factor of* $18x^3y^2$. (The lowest exponent of 2 is 1.)

b. 3 is the highest power of 3 which is a common factor, because only one 3 is a factor of $12x^2$ and of $24x^5y$. (The lowest exponent of 3 is 1.)

c. x^2 is the highest power of x which is a common factor, because *only* x^2 is a factor of $12x^2$. (The lowest exponent of x is 2.)

d. y is not a common factor, as it does not appear in $12x^2$.

3. $\therefore$ the H. C. F. is $2 \cdot 3 \cdot x^2$ or $6x^2$.

Check. —

$$\left.\begin{aligned} 12x^2 \div 6x^2 &= 2 \\ 18x^3y^2 \div 6x^2 &= 3xy^2 \\ 24x^5y \div 6x^2 &= 4x^3y \end{aligned}\right\} \text{2, } 3xy^2\text{, and } 4x^3y \text{ do not have any common factors.}$$

Rule. — To find the H. C. F. of two or more expressions:

1. Find all of the prime factors of each expression.

2. Select the factors common to all of the given expressions, and give each the lowest exponent it has in any of the expressions.

3. Form the product of the common factors selected in Step 2.

NOTE 1. — Step 2 of the solution above should be done mentally.

NOTE 2. — The arithmetical coefficient of the H. C. F. can often be recognized at once, as it is the largest common divisor of the coefficients of the given expressions.

Thus, in this example, 6 is easily recognized as the largest common divisor of 12, 18, and 24.

Example 2. — *a.* Find the H. C. F. of $3a^3 - 3ab^2$ and $3a^4 - 6a^3b + 3a^2b^2$.

Solution. — 1. Factor each expression.

a. $3a^3 - 3ab^2 = 3a(a^2 - b^2) = 3a(a + b)(a - b).$

b. $3a^4 - 6a^3b + 3a^2b^2 = 3a^2(a^2 - 2ab + b^2) = 3a^2(a - b)(a - b).$

2. $\therefore$ the H. C. F. $= 3 \cdot a(a - b)$, or $3a(a - b)$.

Check. — 1. $3a(a + b)(a - b) \div 3a(a - b) = a + b.$

Because, $3a \div 3a = 1$; $(a - b) \div (a - b) = 1$; and $(a + b)$ remains in the dividend.

2. $3a^2(a - b)(a - b) \div 3a(a - b) = a(a - b).$

Because, $3a^2 \div 3a = a$; $(a - b) \div (a - b) = 1$.

3. $a + b$ and $a(a - b)$ do not have any common factor.

EXERCISE 81

Find the highest common factor of:

1. $18a$ and $27a^2$
2. $12b$ and $20bc$
3. $10x^2$ and $25xy^2$
4. $3m^2$ and $7m^3$
5. $14ab^2$ and $22ab^3$
6. $8x^2y$, $10x^3y^2$, and $14x^4y^3$
7. $6m^2n^4$, $15mn^3$, and $24m^3n^2$
8. $7r^2s^3$, $21rs^4$, and $49r^3s^2$
9. $16c^4d^4$, $40c^3d^5$, and $64c^5d^6$
10. $24x^2y^3$, $48xy^2$, and $96x^3y^4$
11. $(x + y)(x + y)$, and $3x(x + y)$
12. $2a(r - s)(r - s)$, and $4a(r - s)(r + s)$
13. $8c(c^2 + d^2)(c + d)$, and $4c^2(c + d)(c - d)$
14. $15(a + 2b)(a - 3b)$, and $9(a - 3b)(a + 3b)$
15. $(m + n)^2(m - n)$, and $(m - n)^2(m + n)$
16. $x^2 - y^2$, and $x^2 + 2xy + y^2$

Hint. — Read Step 1 of the Rule. Examine Example 2.

17. $x^2 - 2x - 35$, and $x^2 + 3x - 10$
18. $6x^2 - x - 2$, and $3x^2 + 7x - 6$
19. $2c^2 + 3cd - 5d^2$, and $2c^2 + 7cd + 5d^2$
20. $x^4 + 3x^3 - 18x^2$, and $x^3 - 7x^2 + 12x$
21. $3x^2 - 15xy + 18y^2$, and $24x^2 - 60xy - 36y^2$
22. $y^2 - y - 12$, $y^2 - 6y + 8$, and $y^2 + 3y - 28$
23. $a^4 - b^4$, $a^4 + 2a^2b^2 + b^4$, and $3a^3b^2 + 3ab^3$
24. $x^2 + 2xy + y^2$, $3x^3 + 6x^2y + 3xy^2$, and $5x^2y + 10xy^2 + 5y^3$
25. $5a^3b + 5a^2b^2 - 10ab^3$, and $10a^4 + 30a^3b + 20a^2b^2$

IX. FRACTIONS

111. The expression $\frac{a}{b}$ is called a **fraction**; it is read "*a* divided by *b*." The dividend, *a*, is the **numerator**, and the divisor, *b*, is the **denominator** of the fraction; together, they are called the **terms** of the fraction.

The denominator, *b*, must never be zero. (See § 68.)

REDUCTION OF FRACTIONS TO LOWEST TERMS

112. A fraction is in its lowest terms when its numerator and denominator do not have any common factor, except unity.

Thus: *a.* $\frac{2}{3}, \frac{a}{b}, \frac{x+y}{x-y}$ are in their lowest terms.

b. $\frac{3}{12}, \frac{2a^2}{ab}, \frac{(x+y)^2}{(x+y)(x-y)}$ are not in their lowest terms.

113. The first fundamental principle of fractions is used when reducing fractions to lowest terms.

If both terms of $\frac{24}{8}$ be divided by 4, $\frac{6}{2}$ is obtained. $\frac{24}{8} = 3$, and also $\frac{6}{2} = 3$. The value of $\frac{24}{8}$, therefore, is not changed; *only the form or appearance of it is changed.*

Rule. — If the numerator and the denominator of a fraction are both divided by the same number, the value of the fraction is not changed.

Example 1.—Reduce $\frac{3a^2x^2 - 3a^2y^2}{6ax^2 - 12axy + 6ay^2}$ to lowest terms.

Solution. — 1. $\frac{3a^2x^2 - 3a^2y^2}{6ax^2 - 12axy + 6ay^2} = \frac{3a^2(x^2 - y^2)}{6a(x^2 - 2xy + y^2)}$

2. $= \frac{3a^2(x+y)(x-y)}{6a(x-y)(x-y)}.$

3. *a*. Divide 3 and 6 each by 3, and write the quotients as below. Similarly, divide a^2 and a each by a; $(x-y)$ of the numerator and $(x-y)$ of the denominator each by $(x-y)$.

b. Then, $\dfrac{3a^2x^2-3a^2y^2}{6ax^2-12axy+6ay^2}=\dfrac{\overset{1}{\cancel{3}}\,\overset{a}{\cancel{a^2}}(x+y)\overset{1}{\cancel{(x-y)}}}{\underset{2}{\cancel{6}}\,\underset{1}{\cancel{a}}(x-y)\underset{1}{\cancel{(x-y)}}}$

4. $\therefore \qquad = \dfrac{a(x+y)}{2(x-y)}.$

Check. — Let $a=1$, $x=2$, $y=1$.

$$\frac{3a^2x^2-3a^2y^2}{6ax^2-12axy+6ay^2}=\frac{3\cdot1\cdot4-3\cdot1\cdot1}{6\cdot1\cdot4-12\cdot1\cdot2\cdot1+6\cdot1\cdot1}$$

$$=\frac{12-3}{24-24+6}=\frac{9}{6}=\frac{3}{2}.$$

The result $\dfrac{a(x+y)}{2(x-y)}=\dfrac{1\cdot(2+1)}{2(2-1)}=\dfrac{1\cdot3}{2\cdot1}=\dfrac{3}{2}.$

Since the two results are equal ($\frac{3}{2}$ in both cases), the solution is probably correct.

Note 1. — Part *a*, of Step 3, should be done mentally; only part *b* should be written.

Note 2. — The process of crossing out common factors from numerator and denominator has long been called *cancellation*. *The word is unnecessary.* Remember that you are dividing numerator and denominator by some number and are crossing out the factors divided.

Note 3. — The result $\dfrac{a(x+y)}{2(x-y)}$ is the simplest form of the given fraction, and equals the given fraction for all values of a, x, and y. If, therefore, we want the value of the given fraction for any values of a, x, and y, we should substitute in $\dfrac{a(x+y)}{2(x-y)}$, not in the given fraction, because the computation will be easier and the result the same.

Thus, the value of the given fraction for $a=6$, $x=15$, and $y=5$ is $\dfrac{6(15+5)}{15-5}$; that is $\dfrac{6\cdot\overset{2}{\cancel{20}}}{\cancel{10}}$ or 12. It is easier to get the result in this way than to substitute these values in the fraction $\dfrac{3a^2x^2-3a^2y^2}{6ax^2-12axy+6ay^2}$.

Rule. — To reduce a fraction to lowest terms:

1. Factor completely the numerator and denominator.

2. Divide numerator and denominator by all their common factors, and write the quotients, including any 1's obtained as quotients.

3. For the numerator of the result, take the product of the numerator quotients; for the denominator, the product of the denominator quotients.

Example 2. — Reduce $\dfrac{3\,m^2x^2 - 3\,m^2y^2}{3\,m^2(x-y)(x+y)}$ to lowest terms.

Solution. — 1. $$\frac{3\,m^2x^2 - 3\,m^2y^2}{3\,m^2(x-y)(x+y)} = \frac{3\,m^2(x^2-y^2)}{3\,m^2(x-y)(x+y)}$$

2. $$= \frac{\overset{1}{\cancel{3}}\,\overset{1}{\cancel{m^2}}\overset{1}{\cancel{(x-y)}}\overset{1}{\cancel{(x+y)}}}{\underset{1}{\cancel{3}}\,\underset{1}{\cancel{m^2}}\underset{1}{\cancel{(x-y)}}\underset{1}{\cancel{(x+y)}}} = \frac{1}{1} \text{ or } 1.$$

Note 1. — Do not fail to write 1 when you get it as quotient. It helps to avoid a certain error often made by pupils.

Note 2. — The result 1 means that the given fraction has the value 1, no matter what may be the values of m, x, and y. Thus, if $m = 96$, $x = 2$, and $y = -37$, the value of the fraction is 1. **We know it without substituting** because the fraction in its lowest terms is 1.

114. **A serious error in reducing fractions** is illustrated by the solution below, written by a pupil.

$$\frac{2\,a+b}{ac} = \frac{2\,\overset{1}{\cancel{a}}+b}{\cancel{a}c} = \frac{2+b}{c}.$$

To prove that the result is wrong, let $a = 2$, $b = 4$, and $c = 3$.

$$\frac{2\,a+b}{ac} = \frac{4+4}{6} = \frac{8}{6} \text{ or } 1\tfrac{1}{3};\quad \frac{2+b}{c} = \frac{2+4}{3} = \frac{6}{3} \text{ or } 2.$$

The error occurred when a of the numerator was divided by a. **The numerator can be divided only by factors of the numerator**, and a is not a factor of the numerator. It is a factor only of one term of the numerator.

EXERCISE 82

Reduce to lowest terms:

1. $\frac{16}{30}$

2. $\frac{24}{44}$

3. $\frac{32}{72}$

4. $\frac{7m^2}{16mn}$

5. $\frac{15r^2s}{24rs^2}$

6. $\frac{12x^3y^2}{34x^2y^4}$

7. $\frac{12a^2b^2c^3(x-y)}{18ab^2c(x-y)}$

8. $\frac{26m(x+y)(x-y)}{36m^2(x+y)}$

9. $\frac{(r-s)(r+2s)}{(r-s)^2(r+s)}$

10. $\frac{x^2+3x-18}{x^2-8x+15}$

11. $\frac{a^2-7a-18}{a^2+3a+2}$

12. $\frac{5c^2+15c-20}{c^2+6c-7}$

13. $\frac{2x^2-2x-24}{5x^2-25x-20}$

14. $\frac{ax^2-3axy+2ay^2}{bx^2-by^2}$

15. $\frac{m^2-7mn+12n^2}{m^2-9mn+20n^2}$

16. $\frac{5x^2-45y^2}{4x^2+4xy-24y^2}$

17. $\frac{6a^2-a-1}{6a^2-5a+1}$

18. $\frac{8c^2-10cd-3d^2}{12c^2-5cd-2d^2}$

19. $\frac{x^4-y^4}{x^2-2xy+y^2}$

20. $\frac{x^4-y^4}{x^4+2x^2y^2+y^4}$

21. $\frac{2ax^2+7ax-15a}{4cx^2+19cx-5c}$

22. $\frac{12a^2-60a+75}{8a^2+36a-140}$

23. $\frac{2x^2+12xy-32y^2}{5x^2-20y^2}$

24. $\frac{b^2c^2-121b^2d^2}{c^3+14c^2d+33cd^2}$

25. $\frac{3mx^2-2mx-16m}{3x^2+12x+12}$

26. $\frac{x^4-16}{x^4-8x^2+16}$

27. $\frac{x^2-a^2}{(x+a)(x^2+ax-2a^2)}$

28. $\frac{3m^4-3m^3n-18m^2n^2}{2m^2n^2-18n^4}$

29. $\frac{20x^3-125xy^2}{20x^2y+42xy^2-20y^3}$

30. $\frac{4x^3-4xy^2}{(4x^2+4xy)(x-y)}$

31. $\frac{(2a-2b)(a^2+4ab+3b^2)}{(2a^2-2b^2)(a+3b)}$

MULTIPLICATION OF FRACTIONS

115. *In arithmetic,* the product of two fractions is the product of their numerators divided by the product of their denominators.

Thus, $\frac{3}{5} \times \frac{2}{11} = \frac{6}{55}$.

In algebra, the same rule is followed.

Example 1. — Find $\frac{5\,a}{2\,x} \cdot \frac{3\,x^2}{10\,a^2}$.

Solution. — 1.
$$\frac{5\,a}{2\,x} \cdot \frac{3\,x^2}{10\,a^2} = \frac{15\,ax^2}{20\,a^2x}.$$

2. Reducing to lowest terms:
$$\frac{15\,ax^2}{20\,a^2x} = \frac{\overset{3}{\cancel{15}}\,\overset{1}{\cancel{a}}\overset{x}{\cancel{x^2}}}{\underset{4}{\cancel{20}}\,\underset{a}{\cancel{a^2}}\underset{1}{\cancel{x}}} = \frac{3\,x}{4\,a}.$$

However, it is preferable to remove any factors common to numerator and denominator by dividing both numerator and denominator by factors common to them before multiplying.

Thus,
$$\frac{5\,a}{2\,x} \cdot \frac{3\,x^2}{10\,a^2} = \frac{\overset{1}{\cancel{5}}\,\cancel{a}}{2\,\cancel{x}} \cdot \frac{3\,\overset{x}{\cancel{x^2}}}{\underset{2\,a}{\cancel{10}\,\cancel{a^2}}} = \frac{3\,x}{4\,a}.$$

Rule. — To find the product of two or more fractions:

1. Find all of the prime factors of the numerators and denominators.

2. Divide out factors common to a numerator and a denominator.

3. Multiply the remaining factors of the numerators for the numerator of the product, and of the denominators for the denominator of the product.

Example 2. — Find $\frac{a^2 + 3\,a - 18}{a^4 - 5\,a^3 + 6\,a^2} \cdot \frac{2\,a^3 - 4\,a^2}{a^2 - 36}$.

Solution. — 1.
$$\frac{a^2 + 3\,a - 18}{a^4 - 5\,a^3 + 6\,a^2} \cdot \frac{2\,a^3 - 4\,a^2}{a^2 - 36} = \frac{(a + 6)(a - 3)}{a^2(a^2 - 5\,a + 6)} \cdot \frac{2\,a^2(a - 2)}{(a - 6)(a + 6)}.$$

2. $$=\frac{\overset{1}{\cancel{(a+6)}}\overset{1}{\cancel{(a-3)}}}{\underset{1}{\cancel{a^2}}\underset{1}{\cancel{(a-3)}}\underset{1}{\cancel{(a-2)}}}\cdot\frac{2\,\overset{1}{\cancel{a^2}}\overset{1}{\cancel{(a-2)}}}{(a-6)\underset{1}{\cancel{(a+6)}}}$$

3. $$=\frac{2}{a-6}.$$

Check. — Let $a = 1$.

$$\frac{a^2+3a-18}{a^4-5a^3+6a^2}\cdot\frac{2a^3-4a^2}{a^2-36}=\frac{1+3-18}{1-5+6}\cdot\frac{2-4}{1-36}$$

$$=\frac{\overset{2}{-\cancel{14}}}{\underset{1}{+\cancel{2}}}\cdot\frac{\overset{1}{-\cancel{2}}}{\underset{5}{-\cancel{35}}}$$

$$=\frac{+2}{-5},\text{ or }-\frac{2}{5}.$$

$$\frac{2}{a-6}=\frac{2}{1-6}$$

$$=\frac{2}{-5}$$

$$=-\frac{2}{5}.$$

NOTE. — **The final result equals the indicated product for all values of a, except any which make a denominator zero.** For example, a cannot be 6 in this example, because when $a = 6$, $a^2 - 36$ is zero and the second fraction does not have any meaning then.

EXERCISE 83

Find the following indicated products:

1. $\frac{5}{6}\cdot\frac{14}{5}$

2. $\frac{3}{8}\cdot\frac{20}{15}$

3. $\frac{5}{2}\cdot\frac{6}{33}$

4. $\frac{3a}{4b}\cdot\frac{2b}{5a}$

5. $\frac{6x}{5y^2}\cdot\frac{10y}{3x^2}$

6. $\frac{3}{10}\cdot\frac{14}{33}\cdot\frac{11}{7}$

7. $\frac{4}{5}\cdot\frac{15}{14}\cdot\frac{7}{6}$

8. $\frac{6}{21}\cdot\frac{14}{5}\cdot\frac{10}{4}$

9. $\frac{2a}{3b}\cdot\frac{6b}{7c}\cdot\frac{5c}{4a}$

10. $\frac{2ab}{3c^2}\cdot\frac{3bc}{4a^2}\cdot\frac{5ac}{6b^2}$

11. $\frac{2xy^2}{3x^3}\cdot\frac{3xy}{y^2}$

12. $\frac{4a^3b}{5x^2}\cdot\frac{15x^2y}{4a^2b}$

13. $\frac{5a^2}{4bc}\cdot\frac{6bc^2}{10a^2bc}$

14. $\frac{27x^3y^2}{20x^4z}\cdot\frac{30xz^2}{18y^3}$

15. $\frac{10m^2n^3}{6nr^3}\cdot\frac{3n^2r^3}{5m^4n^2}$

16. $\frac{x+y}{2x}\cdot\frac{3x^2}{(x+y)^2}$

17. $\frac{a^2-b^2}{4a^2}\cdot\frac{5a^3}{(a+b)^2}$

18. $\frac{6m+12}{3m-9}\cdot\frac{5m-15}{4m+8}$

19. $\frac{am+ax}{dm-dx}\cdot\frac{cm-cx}{bm+bx}$

20. $\dfrac{a^2 - b^2}{(a + b)^2} \cdot \dfrac{ac + bc}{ac - bc}$

22. $\dfrac{x^2 - 16}{(x - 2)^2} \cdot \dfrac{x^2 - 7x + 10}{x^2 - 9x + 20}$

21. $\dfrac{2t + 8}{t^2 - 4} \cdot \dfrac{5t - 10}{t^2 + 6t + 8}$

23. $\dfrac{3y - 15}{2y + 8} \cdot \dfrac{y^2 + 9y + 8}{y^2 + 3y - 40}$

24. $\dfrac{x^2 - y^2}{x^2 - 4y^2} \cdot \dfrac{4x - 8y}{3x^2 + 3xy - 6y^2}$

25. $\dfrac{a^2x - a^3}{x^3 - 3ax^2} \cdot \dfrac{x^2 - ax - 6a^2}{x^2 + ax - 2a^2}$

26. $\dfrac{a^2 - 6a - 27}{a^2 + 9a + 18} \cdot \dfrac{a^2 + 10a + 24}{a^2 - 7a - 18}$

27. $\dfrac{2r^2 + 5rs - 12s^2}{2r^2 - 5rs + 3s^2} \cdot \dfrac{r^2 - 2rs + s^2}{r^2 + 5rs + 4s^2}$

28. $\dfrac{3a - 6b}{9a^2 - b^2} \cdot \dfrac{3a^2 + 2ab - b^2}{a^2 - 4b^2} \cdot \dfrac{a^2 - ab - 6b^2}{6a + 6b}$

29. $\dfrac{a^4 - b^4}{3a - 3b} \cdot \dfrac{4a^2 - b^2}{a^4 + 2a^2b^2 + b^4} \cdot \dfrac{6a^2 + 6b^2}{2a^2 + ab - b^2}$

30. $\dfrac{4x^2 + 12x + 9}{3x^2 + 17x - 28} \cdot \dfrac{3x^2 + 11x - 20}{2x^2 - x - 6} \cdot \dfrac{3x^2 + 22x + 7}{2x^2 + 13x + 15}$

31. $\dfrac{3m^2 - m}{4m^2 - 25} \cdot \dfrac{2m^2 + 15m + 25}{6m} \cdot \dfrac{6m^2 - 15m}{3m^2 + 14m - 5}$

32. $\dfrac{4 - x^2}{12 - 3x} \cdot \dfrac{20 - x - x^2}{4 - 4x + x^2} \cdot \dfrac{6 - 2x}{10 + 7x + x^2}$

33. *a.* $\dfrac{x^2 - mx + m^2}{2x^2 - 2m^2} \cdot \dfrac{2x^2 - 2mx}{5x^2 - 15mx} \cdot \dfrac{x^2 - 2mx - 3m^2}{3x^2 - 3mx + 3m^2}$

b. What is the value of the indicated product in part *a* when $x = 3$ and $m = 2$?

34. $\dfrac{4x^2 - 25y^2}{10x - 20y} \cdot \dfrac{15x^2 - 55xy - 20y^2}{2x^2 - 3xy - 20y^2} \cdot \dfrac{2x - 4y}{6x^2 - 13xy - 5y^2}$

DIVISION OF FRACTIONS

116. *In arithmetic,* to divide one fraction by another, we multiply the dividend by the inverted divisor.

Thus, *a.*
$$\frac{9}{10} \div \frac{3}{5} = \frac{\overset{3}{\cancel{9}}}{\underset{2}{\cancel{10}}} \times \frac{\cancel{5}}{\underset{1}{\cancel{3}}} = \frac{3}{2}.$$

b.
$$3\tfrac{2}{3} \div 2\tfrac{1}{5} = \frac{11}{3} \div \frac{11}{5} = \frac{\overset{1}{\cancel{11}}}{3} \times \frac{5}{\underset{1}{\cancel{11}}} = \frac{5}{3}.$$

In algebra, we do the same.

Rule. — To divide one fraction by another:

1. **Factor the numerators and denominators of the fractions.**
2. **Invert the divisor fraction.**
3. **Multiply the dividend fraction by the inverted divisor.**

Note. — **The divisor follows the sign of division.**

Example. — Divide $\dfrac{a^2m + 10\,am + 21\,m}{a^3 - 4\,a^2 + 3\,a}$ by $\dfrac{a^2m^3 - 9\,m^3}{a^3 - a^2}$.

Solution. — 1. $\dfrac{a^2m + 10\,am + 21\,m}{a^3 - 4\,a^2 + 3\,a} \div \dfrac{a^2m^3 - 9\,m^3}{a^3 - a^2}$

2. $= \dfrac{m(a^2 + 10\,a + 21)}{a(a^2 - 4\,a + 3)} \div \dfrac{m^3(a^2 - 9)}{a^2(a - 1)}$

3. $= \dfrac{\overset{1}{\cancel{m}}(a + 7)\overset{1}{\cancel{(a + 3)}}}{\underset{1}{\cancel{a}}(a - 3)\underset{1}{\cancel{(a - 1)}}} \cdot \dfrac{\overset{a}{\cancel{a^2}}\overset{1}{\cancel{(a - 1)}}}{\underset{m^2}{\cancel{m^3}}(a - 3)\underset{1}{\cancel{(a + 3)}}}$

4. $= \dfrac{a(a + 7)}{m^2(a - 3)^2}.$

Check. — Let $a = 2$; $m = 1$.

$$\frac{a^2m + 10\,am + 21\,m}{a^3 - 4\,a^2 + 3\,a} \div \frac{a^2m^3 - 9\,m^3}{a^3 - a^2} = \frac{4 \cdot 1 + 10 \cdot 2 \cdot 1 + 21}{8 - 16 + 6} \div \frac{4 - 9}{8 - 4}$$

$$= \frac{45}{-2} \div \frac{-5}{4} = \left(-\frac{\overset{9}{\cancel{45}}}{\underset{1}{\cancel{2}}}\right) \times \left(-\frac{\overset{2}{\cancel{4}}}{\underset{1}{\cancel{5}}}\right) = +18.$$

Also, $\dfrac{a(a + 7)}{m^2(a - 3)^2} = \dfrac{2(2 + 7)}{1(2 - 3)^2} = \dfrac{2 \cdot 9}{1(-1)^2} = \dfrac{18}{1} = 18.$

EXERCISE 84

1. $\frac{3}{14} \div \frac{2}{7}$
2. $\frac{9a}{4b} \div \frac{15a}{8b}$
3. $\frac{18x^2}{5y^2} \div \frac{3x}{10y}$
4. $\frac{12c^2d}{28cd^2} \div \frac{4d}{7c}$
5. $\frac{36m^2}{35n^3} \div \frac{9m^3}{14n^2}$
6. $24x^2y^3 \div \frac{3xy^2}{4}$
7. $\frac{5m}{x^2 - 8x + 15} \div \frac{10m}{x^2 - 2x - 15}$
8. $\frac{2c}{4c^2 - 1} \div \frac{4c^2}{6c + 3}$
9. $\frac{5a + 15}{a^2 - 9} \div \frac{5a - 15}{a^2 - 6a + 9}$
10. $\frac{c^2 + d^2}{c + d} \div \frac{c^4 - d^4}{c^2 - 2cd + d^2}$
11. $\frac{12 - 4x}{10 + 5x} \div \frac{9 - 6x + x^2}{6 + 5x + x^2}$
12. $\frac{x^2 - xy}{3x - 6y} \div \frac{x^2 - y^2}{x^2 - xy - 2y^2}$
13. $\frac{a^3 + ab^2}{a^3 - 2ab^2} \div \frac{a^2 + b^2}{a^2 - 4ab + 4b^2}$
14. $\frac{(2r + s)^2}{8r - 4s} \div \frac{4r^3 - rs^2}{8rs - 4s^2}$
15. $\frac{x^2 - x - 42}{x^2 + 5x - 84} \div \frac{x^2 + 2x - 8}{x^2 + 10x - 24}$
16. $\frac{m^3 - 5m^2 - 6m}{3m^2 + 9m + 6} \div \frac{2m^2 - 14m + 12}{m^2 + m - 2}$
17. $\frac{c^2 - 5cd + 6d^2}{c^2 - 9cd + 14d^2} \div \frac{c^2 + 2cd - 15d^2}{c^2 - 2cd - 35d^2}$
18. $\frac{12a^2 - 17ab - 5b^2}{9a^2 - 30ab + 25b^2} \div \frac{8a^2 - 2ab - b^2}{9a^2 - 18ab + 5b^2}$
19. $\frac{a^2 - 16}{a^2 - 9a - 5} \cdot \frac{a^2 - 16a + 55}{a^2 - 7a - 44} \div \frac{3a - 12}{4a^2 - 1}$
20. $\frac{(2x - y)^2}{3x^2 + 15xy} \cdot \frac{4xy - 12y^2}{12x^2 - 4xy - y^2} \div \frac{2x^2 - 7xy + 3y^2}{6x^2 + 31xy + 5y^2}$
21. $\left(\frac{c^2 + d^2}{c^2 - d^2} \div \frac{c^4 + 2c^2d^2 + d^4}{c^2 + 2cd + d^2}\right) \cdot \frac{c^3 + cd^2}{cd + d^2}$

ADDITION AND SUBTRACTION OF FRACTIONS

117. *In arithmetic,* fractions having a common denominator are added or subtracted without difficulty:

Thus, *a.* $\frac{7}{15}+\frac{4}{15}=\frac{11}{15}$. *b.* $\frac{9}{16}-\frac{5}{16}=\frac{4}{16}$.

Fractions which do not have the same denominator must be changed to equal fractions which have a common denominator.

Thus, $\frac{2}{3}+\frac{1}{2}+\frac{1}{4}=\frac{8}{12}+\frac{6}{12}+\frac{3}{12}$ or $\frac{17}{12}$.

12, *the lowest common denominator,* is the smallest number which contains 3, 2, and 4 exactly as divisors.

In algebra, also, before fractions can be added or subtracted, they must be changed to equal fractions having a common denominator.

You must, therefore, learn:

a. how to form the lowest common denominator;

b. how to change fractions to "equal" fractions having this common denominator.

118. **Lowest common multiple.**

a. A **multiple** of a number is any number which contains the given number as an exact divisor.

Thus, 6, 9, 12, and 30 are multiples of 3.

b. A **common multiple of two or more numbers** is a multiple of each of them; *it can be divided by each of them.*

Thus, 24 is a common multiple of 2, 3, 8, and 12.
Also, 48, 72, 96, 120, etc., are common multiples of 2, 3, 8, and 12.

c. The **lowest common multiple of two or more numbers** is the smallest of their common multiples.

Thus, 24 is the lowest common multiple of 2, 3, 8, and 12.

d. The **lowest common multiple of two or more expressions** is that multiple of them which has the least number of prime factors.

Example. — What is the lowest common multiple of $5a^3b$ and $2ab^5c$?

Solution. — 1. The smallest number which will contain both 5 and 2 is 10, — namely, $5 \cdot 2$.

2. The lowest power of a which will contain a^3 and a (separately) is a^3; that is, $a^3 \div a^3 = 1$; $a^3 \div a = a^2$.

3. The lowest power of b which will contain b and b^5 is b^5.

4. The lowest power of c which will contain c is c.

5. $\therefore$ the L. C. M. $= 10a^3b^5c$.

Check. — Does $10a^3b^5c$ contain each of the expressions?

Yes, for $\dfrac{10a^3b^5c}{5a^3b} = 2b^4c$; and $\dfrac{10a^3b^5c}{2ab^5c} = 5a^2$.

Rule. — To find the L. C. M. of two or more expressions:

1. Find the prime factors of each of the expressions.

2. Select all of the different prime factors and give to each the highest exponent with which it appears in the given expressions.

3. Form the product of all of the factors selected in Step 2.

Example. — Find the L. C. M. of $2am^2(m + a)$ and $9a^2(m + a)^3$.

Solution. — 1. $2am^2(m+a) = 2 \cdot a \cdot m^2 \cdot (m+a)$.

$9a^2(m+a)^3 = 3^2 \cdot a^2 \cdot (m+a)^3$.

2. 2 appears with 1 as its highest exponent.

3 appears with 2 as its highest exponent.

a appears with 2 as its highest exponent.

m appears with 2 as its highest exponent.

$(m + a)$ appears with 3 as its highest exponent.

$$\therefore \text{L. C. M.} = 2 \cdot 3^2 \cdot a^2 \cdot m^2 \cdot (m+a)^3 = 18a^2m^2(m+a)^3.$$

Check. — Does $18a^2m^2(m+a)^3$ contain each of the given expressions?

Yes, for $\dfrac{18a^2m^2(m+a)^3}{2am^2(m+a)} = 9a(m+a)^2$

and $\dfrac{18a^2m^2(m+a)^3}{9a^2(m+a)^3} = 2m^2$.

EXERCISE 85

Find the L. C. M. of the numbers in each example, and check:

1. 6 and 8
2. 12 and 15
3. 18 and 12
4. $6a$ and $9b$
5. $4x^2$ and $10y^2$
6. $4ab$ and $6a^2b$
7. $8mn$ and $12n^2$
8. $4r^2$ and $14r^3$
9. $16r^4$ and $12rs^3$
10. $30m^3$ and $18mn^4$
11. $5a^2$, $10b^2$, and $15c^2$
12. $4c^3$, $6c^2d$, and $10cd^2$
13. $20x$, $15xy$, and $12x^2y$
14. $12y^3$, $8x^2y$, and $15x^3$
15. $27c^4$, $6c^3d$, and $18c^2d^2$
16. $(x+2)(x+5)$ and $(x+2)(x-3)$
17. $(m-2n)^2$ and $(m+n)$
18. $(1+a)^3$, $(1-a)(1+a)$, and $(1+a)^2$
19. $3(a+b)^2$, $4(a-b)(a+b)$, and $2(a-b)^2$
20. $2x-10$ and $3x-12$
21. $3a-7$ and $6a+15$
22. $5a+5b$ and a^2-b^2
23. $4p^2-1$ and $2p+1$
24. x^2-y^2 and $x^2-2xy+y^2$
25. $mx-my$, $nx+ny$, and x^2-y^2
26. x^2-9 and $x^2-8x-33$
27. $4c^2-25$, $6c-15$, and $8c+20$
28. $a^2+2ab+b^2$ and $2a^2+ab-b^2$
29. $4a^2-9b^2$ and $4a^2+12ab+9b^2$
30. $9mx^2-m$, $3mnx-mn$, and $9nx^2-6nx+n$
31. $x^2-10x+21$, $x^2-3x-28$, and x^2+x-12
32. $6r^2+11rs-10s^2$ and $12r^2+rs-6s^2$
33. $10x^2-x-21$, $15x^2+x-28$, and $6x^2-17x+12$

119. The second fundamental principle of fractions.

If both terms of the fraction $\frac{6}{2}$ are multiplied by 4, the result is $\frac{24}{8}$. Since both fractions have the value 3, they are equal.

Rule. — If the numerator and denominator of a fraction are both multiplied by the same number, the value of the fraction is not changed.

This principle is used in changing fractions to a common denominator.

120. The lowest common denominator (L. C. D.) **of two or more fractions** is the lowest common multiple of their denominators.

Example 1. — Change to their lowest common denominator the fractions $\frac{3}{4}$ and $\frac{5}{6}$.

Solution. — 1. The L. C. D. of these fractions is 12, for 12 is the L. C. M. of 4 and 6. (§ 118.)

2. To change the denominator 4 into 12, it must be multiplied by 3; that is, by $12 \div 4$. Then, in order not to change the value of the fraction, multiply also the numerator by 3.

$$\therefore \frac{3}{4} = \frac{3 \cdot 3}{3 \cdot 4} = \frac{9}{12}.$$

3. For the fraction $\frac{5}{6}$: $12 \div 6 = 2$. Multiply numerator and denominator by 2.

$$\therefore \frac{5}{6} = \frac{2 \cdot 5}{2 \cdot 6} = \frac{10}{12}.$$

The fractions have been changed to $\frac{9}{12}$ and $\frac{10}{12}$. These fractions have the common denominator 12. They have the same value as the fractions $\frac{3}{4}$ and $\frac{5}{6}$, because they were derived by use of the rule in § 119.

In algebra, use the following

Rule. — To change fractions to their lowest common denominator:

1. Find the prime factors of the denominators.

2. Find the L. C. M. of the given denominators; this is the L. C. D.

3. For each fraction, divide the L. C. D. by the given denominator; multiply both numerator and denominator by the quotient.

Example 2. — Change to their lowest common denominator:

$$\frac{4\,a}{a^2-4} \text{ and } \frac{3\,a}{a^2-5\,a+6}.$$

Solution. — **1.** $$\frac{4\,a}{a^2-4}=\frac{4\,a}{(a-2)(a+2)}.$$

2. $$\frac{3\,a}{a^2-5\,a+6}=\frac{3\,a}{(a-2)(a-3)}.$$

3. The L. C. D. is $(a+2)(a-2)(a-3)$.

4. L. C. D. $\div\ (a-2)(a+2)=a-3$.

$$\therefore\ \frac{4\,a}{(a-2)(a+2)}=\frac{4\,a(a-3)}{(a-2)(a+2)(a-3)}.$$

5. L. C. D. $\div\ (a-2)(a-3)=(a+2)$.

$$\therefore\ \frac{3\,a}{(a-2)(a-3)}=\frac{3\,a(a+2)}{(a-2)(a+2)(a-3)}.$$

Check. — The final fractions in Steps 4 and 5 may be changed into the original fractions by dividing out common factors.

Note 1. — In Step 4, to get L. C. D. $\div\ (a-2)(a+2)$, write, *if necessary:*

$$\frac{\cancel{(a+2)}\cancel{(a-2)}(a-3)}{\cancel{(a-2)}\cancel{(a+2)}}=a-3.$$

Similarly, in Step 5:

$$\frac{(a+2)\cancel{(a-2)}\cancel{(a-3)}}{\cancel{(a-2)}\cancel{(a-3)}}=a+2.$$

Try to do this mentally, however.

Note 2. — In Step 4, the fractions $\frac{4\,a}{(a-2)(a+2)}$ and $\frac{4\,a(a-3)}{(a-2)(a+2)(a-3)}$ are "equal."

Really they are identically equal. (See § 76.) Such fractions are called *equivalent fractions*, to distinguish them from fractions (which will be studied soon) which are *equal for some few values of their letters.*

Thus, $\frac{x}{2}=\frac{x+1}{3}$ when $x=2$ but not otherwise.

Remember: **equivalent fractions always have the same value but differ in form.**

EXERCISE 86

Change to equivalent fractions having their L. C. D.:

1. $\frac{5}{8}; \frac{4}{3}$

2. $\frac{3}{5}; \frac{3}{4}$

3. $\frac{a}{6}; \frac{a}{4}$

4. $\frac{m}{8}; \frac{n}{5}$

5. $\frac{2a}{3}; \frac{3b}{4}$

6. $\frac{4x^2}{7}; \frac{3y^2}{2}$

7. $\frac{3}{x}; \frac{4}{y}$

8. $\frac{7}{m}; \frac{8}{n}$

9. $\frac{9}{x^2}; \frac{4}{x}$

10. $\frac{6}{ab}; \frac{7}{b}$

11. $\frac{2}{a}; \frac{4}{b}; \frac{6}{c}$

12. $\frac{2a}{m}; \frac{3b}{m^2}; \frac{4c}{m^3}$

13. $\frac{a}{x}; \frac{b}{x^2}; \frac{c}{x^3}$

14. $\frac{5}{2x}; \frac{3}{4y}; \frac{4}{3z}$

15. $\frac{2a}{mn}; \frac{3b}{nr}; \frac{4c}{mr}$

16. $\frac{x}{x+y}; \frac{y}{x-y}$

17. $\frac{a}{2a-b}; \frac{b}{2a+b}$

18. $\frac{2m}{3m+n}; \frac{3n}{2m-n}$

19. $\frac{3}{2r-6}; \frac{4}{3r-9}$

20. $\frac{a}{5x+10}; \frac{b}{7x-21}$

21. $\frac{5}{x^2-9}; \frac{4}{x+3}$

22. $\frac{3}{4m^2-1}; \frac{2}{6m+3}$

23. $\frac{3r^2}{r^2-25s^2}; \frac{r+s}{r-5s}$

24. $\frac{x^2-y^2}{(2x-3y)^2}; \frac{x+2y}{10x-15y}$

25. $\frac{m}{(2m-3n)^2}; \frac{n}{4m^2-9n^2}$

26. $\frac{3x}{x^2-4x-21}; \frac{5x}{x^2-2x-35}$

27. $\frac{3a}{6a^2-a-1}; \frac{2a}{3a^2+7a+2}; \frac{4a}{2a^2+3a-2}$

28. $\frac{m+1}{m^2-m-6}; \frac{m-4}{m^2-4m+3}; \frac{m+3}{m^2+m-2}$

29. $\frac{r}{r^2-6ar+9a^2}; \frac{a}{r^2+4ar-21a^2}; \frac{ar}{r^2+7ar}$

30. $\frac{2}{3x-6y}; \frac{1}{2x-3y}; \frac{3}{8x-4y}$

121. Adding and subtracting fractions is now possible. (See § 117.)

Rule. — To add or subtract fractions:

1. Reduce them, if necessary, to equivalent fractions having their lowest common denominator.

2. For the numerator of the result, combine the numerators of the resulting fractions, in parentheses, preceding each by the sign of its fraction.

3. For the denominator of the result, write the L. C. D.

4. Simplify the numerator by removing parentheses and combining like terms.

5. Reduce the result to lowest terms.

Example. — Find $\frac{5x-4y}{6} - \frac{7x-2y}{14}$.

Solution. — 1. $\frac{5x-4y}{6} - \frac{7x-2y}{14}$

2. $= \frac{7(5x-4y)}{42} - \frac{3(7x-2y)}{42}$ (Rule, Step 1)

3. $= \frac{7(5x-4y)-3(7x-2y)}{42}$ (Rule, Steps 2, 3)

4. $= \frac{35x-28y-21x+6y}{42}$ (Rule, Step 4)

5. $= \frac{14x-22y}{42}$ (Rule, Step 4)

6. $= \frac{2(7x-11y)}{42}$ (Rule, Step 5)

7. $= \frac{7x-11y}{21}$.

Check. — Let $x = 1$; $y = 1$.

$$\left.\begin{array}{l} \frac{5x-4y}{6} = \frac{5-4}{6} \text{ or } \frac{1}{6}; \\ \frac{7x-2y}{14} = \frac{7-2}{14} \text{ or } \frac{5}{14}; \end{array}\right\} \text{and } \frac{1}{6} - \frac{5}{14} = \frac{7}{42} - \frac{15}{42} = \frac{-8}{42}, \text{ or } \frac{-4}{21}.$$

Also, $\frac{7x-11y}{21} = \frac{7-11}{21} = -\frac{4}{21}$.

NOTE 1. — Use this form of solution. The equality sign in line 2 means that the fractions there are equivalent to the ones above in line 1; etc.

NOTE 2. — After a few examples, omit Step 2; pass at once from Step 1 to Step 3. Always write Step 3. Notice the words "in parentheses" in the second step of the rule.

NOTE 3. — Remember that the result, $\frac{7x - 11y}{21}$, is identically equal to $\frac{5x - 4y}{6} - \frac{7x - 2y}{14}$. Therefore, if we want the value of $\frac{5x - 4y}{6} - \frac{7x - 2y}{14}$ for any values of x and y, we should substitute in $\frac{7x - 11y}{21}$ because that is simpler than substituting in $\frac{5x - 4y}{6} - \frac{7x - 2y}{14}$. Thus, when $x = 2$ and $y = 1$, the given expression in line 1 equals $\frac{7 \cdot 2 - 11 \cdot 1}{21}$, or $\frac{3}{21}$, or $\frac{1}{7}$.

EXERCISE 87

1. $\frac{3}{4} + \frac{5}{6}$
2. $\frac{3}{5} - \frac{7}{15}$
3. $\frac{9}{8} - \frac{2}{3}$
4. $\frac{5a}{9} + \frac{a}{3}$
5. $\frac{5m}{6} + \frac{3m}{5}$
6. $\frac{3x}{2} - \frac{2x}{3}$
7. $\frac{13y}{15} - \frac{5y}{6}$
8. $\frac{2}{3} - \frac{3}{4} + \frac{5}{6}$
9. $\frac{2}{5} - \frac{7}{15} + \frac{1}{3}$
10. $\frac{2x}{3} - \frac{x}{2} + \frac{5x}{6}$
11. $\frac{2t}{7} - \frac{9t}{14} + \frac{t}{2}$
12. $\frac{3w}{5} + \frac{3w}{4} - \frac{3w}{2}$
13. $\frac{a}{3} + \frac{b}{4} + \frac{c}{2}$
14. $\frac{x}{3} - \frac{y}{5} + \frac{z}{6}$
15. $\frac{4a - 3}{8} + \frac{2a + 1}{6}$
16. $\frac{a + 3}{3} + \frac{3a - 2}{5}$
17. $\frac{x - 3y}{4} + \frac{2x + 3y}{14}$
18. $\frac{5c + 4}{12} - \frac{c + 3}{9}$
19. $\frac{2c - 1}{5} - \frac{c + 2}{6}$
20. $\frac{3m + 7}{5} - \frac{2m - 1}{4}$
21. $\frac{4x - 3y}{6} - \frac{5x - 4y}{8}$
22. $\frac{2x + 5}{\ } + \frac{x - 7}{4} - \frac{5x + 2}{6}$

23. $\frac{3a+5}{4} - \frac{5a+3}{6} + \frac{a-6}{8}$

24. $\frac{4x+1}{3} - \frac{6x-1}{9} - \frac{4x+3}{6}$

25. $\frac{6c-4d}{3} - \frac{5c-d}{2} + \frac{4c+3d}{5}$

26. $\frac{3m+2n}{4} + \frac{5m-2n}{9} - \frac{4m+n}{3}$

27. $\frac{2r-s}{5} - \frac{3r+2s}{10} + \frac{4r+3s}{15} - \frac{5r-4s}{20}$

28. Find $\frac{2a+b}{ab} + \frac{2b-c}{bc} - \frac{2c+a}{ac}$.

Solution. — 1. $\frac{2a+b}{ab} + \frac{2b-c}{bc} - \frac{2c+a}{ac}$

2. $= \frac{c(2a+b)}{abc} + \frac{a(2b-c)}{abc} - \frac{b(2c+a)}{abc}$.

(Complete the solution and check it.)

NOTE. — The L. C. D., *abc*, is gotten by the rule on page 168. $\frac{2a+b}{ab}$ is equivalent to $\frac{c(2a+b)}{abc}$.

For, $abc \div ab = c$; $\therefore \frac{2a+b}{ab} = \frac{c(2a+b)}{abc}$. (Rule, page 168)

Similarly for the other fractions.

29. $\frac{4}{5a} + \frac{2}{3a}$

30. $\frac{7}{2m} - \frac{4}{5m}$

31. $\frac{4}{a} + \frac{5}{b}$

32. $\frac{6}{r} - \frac{7}{s}$

33. $\frac{a}{x} + \frac{b}{x}$.

34. $\frac{c}{x} - \frac{d}{y}$

35. $\frac{2}{5x} - \frac{3}{4y}$

36. $\frac{a}{3s} + \frac{b}{7t}$

37. $\frac{m}{rx} + \frac{n}{ry}$

38. $\frac{5}{x^2} - \frac{2}{x^3}$

39. $\frac{2}{a^2} - \frac{3}{ab} + \frac{4}{b^2}$

40. $\frac{1}{2x^2} - \frac{2}{3xy} + \frac{5}{6y^2}$

41. $\frac{2x-1}{3x} + \frac{3x+2}{4x}$

42. $\frac{5x+3y}{x^2y} - \frac{2x+4y}{xy^2}$

43. $\frac{x+1}{2x} - \frac{x^2-1}{3x^2}$

44. Find $\dfrac{3}{4p-6} - \dfrac{4}{3p-12}$.

Solution. — 1. $\dfrac{3}{4p-6} - \dfrac{4}{3p-12}$

2. $= \dfrac{3}{2(p-3)} - \dfrac{4}{3(p-4)}$ (Factoring the denominators)

3. $= \dfrac{9(p-4)}{6(p-3)(p-4)} - \dfrac{8(p-3)}{6(p-3)(p-4)}$. (Step 1, rule, page 171)

(Complete the solution, following Steps 2–5 of the rule.)

NOTE. — In Step 3, to get $\dfrac{9(p-4)}{6(p-3)(p-4)}$:

a. $6(p-3)(p-4) \div 2(p-3) = 3(p-4)$. (See Step 3, rule, page 168.)

b. Multiplying both terms of $\dfrac{3}{2(p-3)}$ by $3(p-4)$, we get $\dfrac{9(p-4)}{6(p-3)(p-4)}$.

45. $\dfrac{1}{m-1} + \dfrac{1}{m+1}$

46. $\dfrac{1}{m-1} - \dfrac{1}{m+1}$

47. $\dfrac{a}{a-b} - \dfrac{b}{a+b}$

48. $\dfrac{x}{x+y} + \dfrac{x}{x-y}$

49. $\dfrac{4}{3x-1} + \dfrac{3}{2x+1}$

50. $\dfrac{6}{5y-3} - \dfrac{2}{4y-1}$

51. $\dfrac{2}{4x-3a} - \dfrac{3}{6x+a}$

52. $\dfrac{4}{3x-3y} - \dfrac{3}{2x+2y}$

53. $\dfrac{3b}{5a+10} + \dfrac{b}{3a-3}$

54. $\dfrac{1}{6x-2y} - \dfrac{5}{30x-9y}$

55. Find $\dfrac{a+b}{4a^2-9b^2} - \dfrac{a-b}{(2a+3b)^2}$.

Solution. — 1. $\dfrac{a+b}{4a^2-9b^2} - \dfrac{a-b}{(2a+3b)^2}$

2. $= \dfrac{a+b}{(2a-3b)(2a+3b)} - \dfrac{a-b}{(2a+3b)(2a+3b)}$

3. $= \dfrac{(a+b)(2a+3b)-(a-b)(2a-3b)}{(2a-3b)(2a+3b)(2a+3b)}$

4. $$= \frac{(2a^2 + 5ab + 3b^2) - (2a^2 - 5ab + 3b^2)}{(2a - 3b)(2a + 3b)(2a + 3b)}.$$

(Complete this solution and check it.)

NOTE 1. — Steps 1, 2, and 3 of the rule on page 171 have been taken at once in Step 3 of this solution. The first fraction of Step 2 is equivalent to $\frac{(a+b)(2a+3b)}{\text{L. C. D.}}$; the second to $\frac{(a-b)(2a-3b)}{\text{L. C. D.}}$. Do this from now on.

NOTE 2. — In Step 4, $(2a^2 + 5ab + 3b^2)$ is $(a + b)(2a + 3b)$, and $(2a^2 - 5ab + 3b^2)$ is $(a - b)(2a - 3b)$. **Always inclose such products in parentheses as in this Step 4.** (Read the preceding sentence three times, — or more; enough to remember it.)

56. $\frac{a-2}{a-3} - \frac{a+2}{a+3}$

57. $\frac{x-2}{(x+5)^2} - \frac{x+2}{x^2-25}$

58. $\frac{3r+s}{3r-s} - \frac{9r^2+s^2}{9r^2-s^2}$

59. $\frac{2m-3}{4m-2} + \frac{m-9}{3m+6}$

60. $\frac{5x-4y}{5x-y} - \frac{3x+4y}{3x+y}$

61. $\frac{r+4}{r^2-3r} - \frac{r+6}{r^2-r-6}$

62. $\frac{3}{4a^2-9} - \frac{2}{6a^2-9a}$

63. $\frac{x}{x^2-6x+8} - \frac{x}{x^2-16}$

64. $\frac{a+b}{2a+6b} - \frac{a-b}{3a+3b}$

65. $\frac{a+b}{x^2-2xy+y^2} - \frac{a-b}{x^2-y^2}$

66. $\frac{a+5}{a^2-a-6} + \frac{a+3}{a^2+7a+10}$

67. $\frac{a+3b}{a^2-7ab+12b^2} - \frac{a-3b}{a^2-ab-12b^2}$

68. $\frac{m+2}{m^2+2m-3} - \frac{m-3}{m^2-3m+2} + \frac{m+1}{m^2+m-6}$

69. $\frac{c+3d}{2c^2+cd-d^2} - \frac{c+2d}{3c^2+2cd-d^2} - \frac{c-d}{6c^2-5cd+d^2}$

70. $\frac{a}{a^2+2ab+b^2} - \frac{1}{a-b} - \frac{b}{a^2-2ab+b^2}$

MIXED EXPRESSIONS

122. **An integral expression** is an expression which does not have literal numbers in the denominators of fractions; as, $a^2 + b^2$, or $\frac{2}{3}a + b$.

An integral expression may be considered a fraction with denominator 1; thus, $a + b$ is the same as $\dfrac{a+b}{1}$.

123. **A mixed expression** consists of one or more integral terms plus or minus one or more fractional terms; as $a^2 + \dfrac{2}{a}$, or $1 + \dfrac{a+b}{a}$.

124. **Changing mixed expressions to fractions** is like adding or subtracting fractions.

Thus, $12\frac{3}{8} = 12 + \dfrac{3}{8} = \dfrac{96+3}{8} = \dfrac{99}{8}$.

Also, $x + \dfrac{x-3}{2} = \dfrac{x}{1} + \dfrac{x-3}{2} = \dfrac{2x + x - 3}{2} = \dfrac{3x-3}{2}$.

EXERCISE 88

Change the following mixed expressions to fractions:

1. $10 + \dfrac{2}{3}$

2. $5 - \dfrac{3}{4}$

3. $\dfrac{3}{2} + 2$

4. $\dfrac{4}{5} - 3$

5. $a + \dfrac{b}{2}$

6. $c - \dfrac{e}{3}$

7. $2 - \dfrac{1}{3} + \dfrac{1}{4}$

8. $\dfrac{3}{5} + 2 - \dfrac{1}{6}$

9. $1 + \dfrac{3}{2} + 5a$

10. $2 - \dfrac{5-2x}{3}$

11. $a - 1 + \dfrac{5}{2a}$

12. $c - 4 + \dfrac{5}{3c}$

13. $1 - \dfrac{2x-3y}{2x+3y}$

14. $3 - \dfrac{a-5x}{a-2x}$

15. $5 - \dfrac{a^2-19x^2}{a^2-4x^2}$

16. $2 + \dfrac{2s^3}{r^3-s^3}$

17. $6 + \dfrac{5x+2}{x^2-1}$

18. $1 - \dfrac{a-16}{a^2-a}$

19. $2+\frac{1}{a}$

20. $3-\frac{2}{b}$

21. $x-\frac{2}{y}$

22. $z+\frac{r}{s}$

23. $x^2-\frac{2x^2}{3}+1$

24. $a-\frac{a^2-1}{a}$

25. $\frac{a}{a-b}-1$

26. $x-\frac{x^2}{x+y}$

27. $1-\frac{2b^3}{a^3+b^3}$

28. $1-\frac{2ab}{a^2+ab+b^2}$

29. $a+b+\frac{b^2}{a-b}$

30. $3a-b-\frac{9a^2-2b^2}{3a+b}$

125. Multiplication and division of mixed expressions.

In such examples, the mixed expressions are inclosed in parentheses. Change the mixed expressions to fractions, doing what the signs direct; then multiply or divide as the signs indicate.

Example. — Find $\left(4-\frac{1}{x+1}\right)\div\left(16+\frac{7}{x^2-1}\right)$.

Solution. — 1. $\left(4-\frac{1}{x+1}\right)\div\left(16+\frac{7}{x^2-1}\right)$

2. $=\left(\frac{4(x+1)-1}{x+1}\right)\div\left(\frac{16(x^2-1)+7}{x^2-1}\right)$

3. $=\frac{4x+4-1}{x+1}\div\frac{16x^2-16+7}{x^2-1}$

4. $=\frac{4x+3}{x+1}\cdot\frac{x^2-1}{16x^2-9}$

5. $=\frac{\overset{1}{\cancel{4x+3}}}{\underset{1}{\cancel{x+1}}}\cdot\frac{(x-1)\overset{1}{\cancel{(x+1)}}}{\underset{1}{\cancel{(4x+3)}}(4x-3)}$

6. $=\frac{x-1}{4x-3}$.

Check. — Let $x=2$.

$\left(4-\frac{1}{x+1}\right)=4-\frac{1}{3}=3\frac{2}{3}$, or $\frac{11}{3}$;

$\left(16+\frac{7}{x^2-1}\right)=16+\frac{7}{3}=18\frac{1}{3}$, or $\frac{55}{3}$;

$\frac{11}{3}\div\frac{55}{3}=\frac{\overset{1}{\cancel{11}}}{\underset{1}{\cancel{3}}}\cdot\frac{\overset{1}{\cancel{3}}}{\underset{5}{\cancel{55}}}=\frac{1}{5}$.

Also, $\frac{x-1}{4x-3}=\frac{2-1}{8-3}=\frac{1}{5}$.

EXERCISE 89

1. $\left(4+\frac{1}{5}\right)\left(8+\frac{1}{3}\right)$

2. $\left(13-\frac{1}{2}\right)\left(2-\frac{4}{15}\right)$

3. $\left(\frac{a}{b}+1\right)\left(\frac{a}{b}-1\right)$

4. $\left(1+\frac{3}{x-2}\right)\left(2-\frac{6}{x+1}\right)$

5. $\left(2+\frac{x}{x-2y}\right)\left(1+\frac{2y}{x-4y}\right)$

6. $\left(a-\frac{b^2}{a}\right)\left(2-\frac{a^2-2b^2}{a^2-b^2}\right)$

7. $\left(\frac{2}{a}-\frac{3}{b}\right)\left(a+\frac{ab}{2b-3a}\right)$

8. $\left(2+\frac{3}{4}\right)\div\left(3-\frac{1}{8}\right)$

9. $\left(6+\frac{2}{3}\right)\div\left(2-\frac{5}{6}\right)$

10. $\left(x-\frac{y^2}{x}\right)\div\left(1+\frac{y}{x}\right)$

11. $\left(3-\frac{1}{x+2}\right)\div\left(3-\frac{4}{x+3}\right)$

12. $\left(a+\frac{b^2}{a-b}\right)\div\left(a+\frac{2ab+b^2}{a-b}\right)$

13. $\left(\frac{y}{x}-\frac{4x}{y}\right)\div\left(\frac{y^2}{4x^2}-\frac{8x^2}{y^2}\right)$

14. $\left(\frac{a}{3b}-\frac{1}{2}\right)\div\left(\frac{1}{6b}-\frac{1}{4a}\right)$

15. $\left(1+\frac{ab}{a^2-ab+b^2}\right)\cdot\left(1+\frac{ab}{a^2+b^2}\right)$

16. $\left(x-4-\frac{21}{x}\right)\div\left(x-8+\frac{7}{x}\right)$

17. $\left(3+\frac{5x-1}{x^2-9}\right)\div\left(3+\frac{2}{x-3}\right)$

18. $\left(1+\frac{2ax+2a^2}{x^2-a^2}\right)\cdot\left(2-\frac{x+4a}{2x+2a}\right)$

19. $\left(2+\frac{7}{m-3}\right)\left(2-\frac{5}{m+2}\right)\div\left(2+\frac{30}{4m^2-16}\right)$

20. $\left(1 - \frac{8x}{x^2 + 4x + 3}\right)\left(2 + \frac{8}{x - 1}\right) \div \left(3 + \frac{6}{x - 5}\right)$

21. $\left(1 - \frac{12y^2}{x^2 + 6xy + 5y^2}\right)\left(1 - \frac{2xy - 4y^2}{x^2 + 5xy - 14y^2}\right) \div \left(1 - \frac{2y}{x + y}\right)$

126. A **complex fraction** is a fraction having one or more fractions in either or both of its terms; as,

$$\frac{a}{2 + \frac{1}{a}} \quad \text{and} \quad \frac{\frac{a}{a-b} - \frac{a}{a+b}}{\frac{b}{a-b} + \frac{a}{a+b}}.$$

Since a fraction is merely an indicated division, complex fractions are merely another way of writing division of mixed expressions.

Example. — Simplify $\dfrac{\frac{a}{a-b} - \frac{a}{a+b}}{\frac{b}{a-b} + \frac{a}{a+b}}$.

Solution. — This means: *a.* Find the difference in the numerator; *b.* find the sum in the denominator; *c.* find the quotient.

The first two of these steps are carried along simultaneously below.

1. $\dfrac{\frac{a}{a-b} - \frac{a}{a+b}}{\frac{b}{a-b} + \frac{a}{a+b}} = \dfrac{\frac{a(a+b) - a(a-b)}{(a-b)(a+b)}}{\frac{b(a+b) + a(a-b)}{(a-b)(a+b)}}$

2. $= \dfrac{\cancel{a^2} + ab - \cancel{a^2} + ab}{\underset{1}{\cancel{(a-b)}}\underset{1}{\cancel{(a+b)}}} \cdot \dfrac{\overset{1}{\cancel{(a-b)}}\overset{1}{\cancel{(a+b)}}}{\cancel{ab} + b^2 + a^2 - \cancel{ab}}$

3. $= \dfrac{2ab}{a^2 + b^2}$

(Check this solution by letting $a = 2$ and $b = 1$.)

NOTE. — A complex fraction can be simplified by multiplying *its* numerator and denominator by the L. C. M. of the denominators of the fractions in its numerator and denominator.

EXERCISE 90

Simplify:

1. $\dfrac{4}{5-\dfrac{1}{5}}$

2. $\dfrac{1-\dfrac{1}{2}}{\dfrac{2}{3}}$

3. $\dfrac{2+\dfrac{3}{5}}{4-\dfrac{1}{10}}$

4. $\dfrac{3+\dfrac{1}{3}}{5-\dfrac{1}{9}}$

5. $\dfrac{\dfrac{3}{8}-1}{1+\dfrac{1}{2}}$

6. $\dfrac{\dfrac{3}{c}}{5-\dfrac{1}{c}}$

7. $\dfrac{\dfrac{7m^2n}{x}}{\dfrac{14m}{x^2}}$

8. $\dfrac{2}{1-\dfrac{1}{x}}$

9. $\dfrac{y-\dfrac{y}{3}}{y+\dfrac{y}{3}}$

10. $\dfrac{x+\dfrac{1}{2}}{x^2-\dfrac{1}{4}}$

11. $\dfrac{2x-\dfrac{1}{3}}{2x+\dfrac{1}{3}}$

12. $\dfrac{\dfrac{2}{3}-y}{\dfrac{2}{3}+y}$

13. $\dfrac{\dfrac{a}{3}-\dfrac{b}{2}}{\dfrac{a}{3}+\dfrac{5b}{6}}$

14. $\dfrac{x+\dfrac{y}{2}}{x^2-\dfrac{y^2}{4}}$

15. $\dfrac{a+\dfrac{b}{c}}{a-\dfrac{b}{c}}$

16. $\dfrac{\dfrac{a}{b}+\dfrac{b}{a}}{\dfrac{a}{b}-\dfrac{b}{a}}$

17. $\dfrac{1+\dfrac{a}{b}}{1-\dfrac{a}{b}+\dfrac{a^2}{b^2}}$

18. $\dfrac{2-\dfrac{3b}{a}}{\dfrac{2a}{3b}-\dfrac{3b}{2a}}$

19. $\dfrac{x-13+\dfrac{40}{x}}{x-3-\dfrac{10}{x}}$

20. $\dfrac{1-\dfrac{4}{y}-\dfrac{21}{y^2}}{\dfrac{2}{y}-\dfrac{14}{y^2}}$

21. $\dfrac{\dfrac{a}{a-b}}{1+\dfrac{b}{a-b}}$

22. $\dfrac{\dfrac{z}{x+y}}{1-\dfrac{z}{x+y}}$

23. $\dfrac{\dfrac{2a}{a+b}-1}{1-\dfrac{a}{a+b}}$

24. $\dfrac{1-\dfrac{4s}{r+s}}{\dfrac{r}{3s}-\dfrac{3s}{r}}$

25. $$\frac{\frac{1}{x}-\frac{1}{y}}{\frac{1}{xy}}$$

26. $$\frac{1+\frac{1}{x}}{1+\frac{2x+1}{x^2}}$$

27. $$\frac{c-\frac{2d^2}{c+d}}{c+\frac{d^2}{c-2d}}$$

28. $$\frac{\frac{x}{x-2}-\frac{x}{x+3}}{\frac{1}{x^2+x-6}}$$

29. $$\frac{\frac{y}{y+2}-\frac{y}{y-3}}{\frac{y^2}{y+2}-\frac{y^2}{y-3}}$$

30. $$\frac{\frac{3}{a-2}-\frac{2}{a-3}}{1-\frac{3}{a-2}}$$

31. $$\frac{2a+\frac{1-2a}{3a}-1}{a-\frac{7}{3}+\frac{2}{3a}}$$

32. $$\frac{x^2+x+1-\frac{1}{x-1}}{1-\frac{1}{x-1}}$$

33. $$\frac{\frac{m}{n}-\frac{m+n}{m-n}}{\frac{m}{n}+\frac{m+n}{m-n}}$$

34. $$\frac{\frac{2}{3x}+\frac{1}{6x^2}-\frac{1}{2x^3}}{2-\frac{1}{2x}-\frac{3}{4x^2}}$$

35. $$\frac{1-\frac{10(x^2-y^2)}{x^2-9y^2}}{1-\frac{2(2x+y)}{x+3y}}$$

SUPPLEMENTARY TOPIC

127. For any fraction, **three signs must be considered,** — the signs of the numerator, of the denominator, and of the fraction itself. Certain rules for making necessary changes in these signs follow.

Rule 1. — If the sign of one term of a fraction is changed, the sign of the fraction also must be changed.

Thus: *a.* $$\frac{-4}{+12}=-\frac{1}{3}; \text{ but } \frac{+4}{+12}=+\frac{1}{3}.$$

In this example, when -4 is changed to $+4$, the value of the fraction changes from $-\frac{1}{3}$ to $+\frac{1}{3}$.

b. $$\frac{-4}{+12} = -\frac{1}{3}; \text{ but } \frac{-4}{-12} = +\frac{1}{3}.$$

In this example, when $+12$ is changed to -12, the value of the fraction changes from $-\frac{1}{3}$ to $+\frac{1}{3}$.

In case it is desired to change the sign of a fraction, the sign of one of the terms of the fraction must also be changed.

Rule 2. — If the signs of both terms of a fraction are changed, the sign of the fraction is not changed.

Thus, $$\frac{-4}{+12} = -\frac{1}{3}; \text{ and also } \frac{+4}{-12} = -\frac{1}{3}.$$

In this example, -4 is changed to $+4$, and $+12$ to -12, but the value of the fraction remains $-\frac{1}{3}$.

The need of these rules is illustrated by the following example.

Example.—Reduce to lowest terms $\dfrac{x^2-9}{12+2x-2x^2}$.

Solution. — 1. The terms of the denominator, not being in the same order as those of the numerator, must be rearranged.

Then $$\frac{x^2-9}{12+2x-2x^2} = \frac{x^2-9}{-2x^2+2x+12}.$$

2. **The negative coefficient of $-2x^2$ is inconvenient.** Change the signs of the denominator by multiplying by -1. Then by Rule 1:

$$\frac{x^2-9}{-2x^2+2x+12} = -\frac{x^2-9}{2x^2-2x-12}$$

3. $$= -\frac{(x-3)(x+3)}{2(x^2-x-6)}$$

4. $$= -\frac{\overset{1}{\cancel{(x-3)}}(x+3)}{2\underset{1}{\cancel{(x-3)}}(x+2)}, \text{ or } -\frac{x+3}{2(x+2)}.$$

Check. — Let $x = 1$.

$$\frac{x^2-9}{12+2x-2x^2} = \frac{1-9}{12+2-2} = \frac{-8}{12} = -\frac{2}{3}.$$

Also, $$-\frac{x+3}{2(x+2)} = -\frac{1+3}{2(1+2)} = -\frac{4}{6} = -\frac{2}{3}.$$

128. **There are two ways in which the signs of a numerator or of a denominator can be changed.**

a. If the expression is not factored, change the sign of every term by multiplying each of them by -1.

Thus, $(-1)(-2x^2-3x+4)=2x^2+3x-4$.

b. If the expression is factored, change the signs of all the terms in an odd number of factors of the expression.

Thus: $2\times(-3)\times 4=-24$, but $2\times(+3)\times 4=+24$.
$2\times(-3)\times 4=-24$, but $(-2)\times(+3)\times(-4)=+24$.

Similarly, consider $(x-y)(y-z)$.
$y-z$ becomes $-y+z$ or $z-y$ when its signs are changed.
Therefore $(x-y)(y-z)=-(x-y)(z-y)$, for changing the signs of $y-z$ changes the sign of the product.
On the other hand, $(x-y)(y-z)=+(y-x)(z-y)$ because the signs of $x-y$ and of $y-z$ have been changed. This means that -1 has been used twice as a factor, and therefore the sign of the product is not changed.

Do not confuse "changing the signs of the terms of an expression" and "rearranging the terms of an expression."

Thus, $2x-3$ becomes $3-2x$ when its signs are changed.
$2x-3$ becomes $-3+2x$ when its terms are rearranged.

NOTE. — Changing the signs of the terms of an expression changes the value of the expression. Rearranging the terms of an expression does not change the value of the expression.

EXERCISE 91

Reduce to lowest terms:

1. $\dfrac{x-5}{15-3x}$
2. $\dfrac{a-2b}{4b^2-a^2}$
3. $\dfrac{y^2-x^2}{4x-4y}$
4. $\dfrac{6+m-m^2}{m^2+m-12}$
5. $\dfrac{15c-10d}{2d^2-cd-3c^2}$
6. $\dfrac{4my-3mx}{9x^2-9xy-4y^2}$

7. Reduce to lowest terms $\dfrac{(x-y)(y-z)}{2\,xyz(z-y)}$.

Solution. — 1. $(y - z)$ differs from $(z - y)$ only in the signs of y and z. Change the signs of the numerator (§ 128, *b*). Therefore, change the sign of the fraction. (Rule 1.)

Then

$$\frac{(x-y)(y-z)}{2\,xyz(z-y)} = -\frac{(x-y)\overset{1}{\cancel{(z-y)}}}{2\,xyz\underset{1}{\cancel{(z-y)}}}$$

$$= -\frac{x-y}{2\,xyz}.$$

2. In case of a result such as that preceding, change the signs of $x - y$, and then change the sign of the fraction.

$$\therefore\ \frac{(x-y)(y-z)}{2\,xyz(z-y)} = -\frac{x-y}{2\,xyz},\ \text{or}\ \frac{y-x}{2\,xyz}.$$

8. $\dfrac{(a-b)(b-c)}{2(c-b)}$

9. $\dfrac{(x-3)(x-4)}{(3-x)(4-x)(5-x)}$

10. $\dfrac{x^2-y^2}{(y-x)^2}$

11. $\dfrac{(c-d)^2}{cd-c^2}$

12. $\dfrac{ab^2+a^2b-2\,a^3}{a^2+ab-2\,b^2}$

13. $\dfrac{(x-y)(x^2+xy+y^2)}{3\,y^2-3\,x^2}$

14. $\dfrac{(a-b)(b-c)(c-a)}{(b-a)(b-c)(a-c)}$

15. $\dfrac{(x-y)^3}{(y-x)^2}$

Find:

16. $\dfrac{x^2-1}{6+x-x^2}\cdot\dfrac{3-x}{x-2}$

17. $\dfrac{x^2-4\,x+4}{6\,x+15}\cdot\dfrac{2\,x+5}{4-x^2}$

18. $\dfrac{x^2-1}{4-2\,x}\cdot\dfrac{4-x^2}{x^2-x-2}\cdot\dfrac{3\,x-6}{2-x-x^2}$

19. $\dfrac{10+5\,m}{8\,m-4}\cdot\dfrac{9-3\,m}{10\,m+5}\div\dfrac{12-3\,m^2}{8\,m^2-2}$

20. $\dfrac{a-2\,a^2}{4\,a^2-1}\cdot\dfrac{1+4\,a+4\,a^2}{5\,a}\cdot\dfrac{10\,a-10\,a^2}{2+2\,a-4\,a^2}$

21. Find $\dfrac{3}{a-b}+\dfrac{2b+a}{b^2-a^2}-\dfrac{2}{b+a}$.

Solution. — 1. It is convenient to have the denominators and numerators arranged in descending powers of a.

Therefore we write:

$$\frac{3}{a-b}+\frac{a+2b}{-a^2+b^2}-\frac{2}{a+b}$$

This has the same value as the original expression, since the terms have been merely rearranged. No signs have been changed yet.

2. In the second fraction, the **denominator** $-a^2+b^2$ **is inconvenient since the coefficient of** $-a^2$ **is** -1. Therefore, change the signs of this denominator, and of the fraction. (Rule 1, page 181.)

$$\therefore \frac{3}{a-b}+\frac{2b+a}{b^2-a^2}-\frac{2}{b+a}=\frac{3}{a-b}-\frac{a+2b}{a^2-b^2}-\frac{2}{a+b}.$$

(Complete this solution.)

22. $\dfrac{2x}{x^2-25}+\dfrac{3}{5-x}$

23. $\dfrac{1}{3x-6}-\dfrac{1}{8-4x}$

24. $\dfrac{2}{6x-9}+\dfrac{x+2}{9-4x^2}$

25. $\dfrac{4}{a^2-5a}-\dfrac{a-13}{10a-2a^2}$

26. $\dfrac{2}{3+c}-\dfrac{3}{3-c}-\dfrac{4c}{c^2-9}$

27. $\dfrac{m+1}{5m^2-7m}+\dfrac{m+2}{7+2m-5m^2}$

28. $\dfrac{x+2}{x-2}+\dfrac{2-x}{2+x}+\dfrac{16}{4-x^2}$

29. $\dfrac{2}{x^2-5x+6}-\dfrac{5}{15-2x-x^2}$

30. $\dfrac{a+3}{3a^2+7a+2}+\dfrac{3a+2}{1+a-6a^2}+\dfrac{2-a}{2-3a-2a^2}$

NOTE. — Additional examples are to be found on page 432.

X. SIMPLE FRACTIONAL EQUATIONS

129. Changing equations which involve fractions to equations which do not involve fractions. You have already done this in examples on page 14. You are now ready to do more difficult ones.

Example. — Solve the equation

$$\frac{3x-1}{4} - \frac{4x-5}{5} = 4 + \frac{7x+5}{10}.$$

Solution. — 1. To eliminate (get rid of) the fractions, multiply by the L. C. M. of the denominators, 20.

$$2.\ \ M_{20}\quad \overset{5}{\cancel{20}} \cdot \frac{(3x-1)}{\cancel{4}} - \overset{4}{\cancel{20}}\ \frac{(4x-5)}{\cancel{5}} = 20 \cdot 4 + \overset{2}{\cancel{20}}\ \frac{(7x+5)}{\cancel{10}}.$$

3. $\therefore\ 5(3x-1) - 4(4x-5) = 80 + 2(7x+5).$

4. $\therefore\ 15x - 5 - 16x + 20 = 80 + 14x + 10.$

5. $\therefore\ 15 - x = 90 + 14x.$

6. $\therefore\ -15x = 75.$

7. $D_{-15}\quad x = -5.$

Check. — Substitute -5 for x in the original equation.

Does $\dfrac{-16}{4} - \dfrac{-25}{5} = 4 + \dfrac{-30}{10}$?

Does $-4 + 5 = 4 - 3$? Yes.

$\therefore\ -5$ is the correct root of the equation.

Note 1. — The fractions **can be eliminated** by multiplying by **any common multiple** of 4, 5, and 10; for example, by 100. **Naturally,** we multiply by the **least** common multiple of 4, 5, 10.

Note 2. — In checking from now on, try to do some of it mentally. Thus, in the check of this solution, -16 is $3(-5) - 1$; -25 is $4(-5) - 5$.

The process of changing the given equation to one which does not have any fractions is called **clearing the equation of fractions.**

While the use of this expression is not necessary, its use is not objectionable *provided you remember that "clearing the equation of fractions"* is accomplished by multiplying both members of the equation by the least common multiple of the denominators.

EXERCISE 92

Solve and check the following equations:

1. $\frac{x}{5} + \frac{3x}{2} = \frac{17}{2}$

2. $\frac{5y}{4} - \frac{7y}{6} = \frac{1}{4}$

3. $\frac{z+5}{3} + 2 = \frac{z+16}{4}$

4. $\frac{w-8}{6} + 3 = \frac{w+12}{7}$

5. $\frac{c-6}{8} + 1 = \frac{2c+4}{9}$

6. $\frac{m+4}{2} = \frac{13}{2} - \frac{m+9}{10}$

7. $\frac{a-5}{4} - \frac{12-a}{6} = \frac{1}{2}$

8. $\frac{t+11}{8} = \frac{7-t}{2} - 4$

9. $\frac{r+16}{6} = \frac{r+4}{3}$

10. $\frac{3t+5}{8} + \frac{2t+3}{4} = -3$

11. $\frac{4x-1}{3} - \frac{x-1}{2} = \frac{7}{12}$

12. $\frac{6t+3}{7} + \frac{3t-5}{3} = 0$

13. $\frac{8z-1}{3} = \frac{z+1}{2} - \frac{11}{8}$

14. $\frac{7y+6}{3} = 4 - \frac{14y+1}{4}$

15. $\frac{12z+2}{11} + \frac{4z+1}{2} = 3$

16. $\frac{s-8}{9} - \frac{s+5}{2} = \frac{1}{2}$

17. $\frac{7x-8}{14} - \frac{7x+6}{4x} = \frac{x-5}{2} - \frac{4x+9}{7x}$

Solution. — 1. The L. C. M. of the denominators is $28x$.

2. M_{28x} $\overset{2x}{\cancel{28x}} \cdot \frac{(7x-8)}{\underset{1}{\cancel{14}}} - \overset{7}{\cancel{28x}} \cdot \frac{(7x+6)}{\cancel{4x}}$

$$= \overset{14x}{\cancel{28x}} \cdot \frac{(x-5)}{\cancel{2}} - \overset{4}{\cancel{28x}} \cdot \frac{(4x+9)}{\cancel{7x}}.$$

3. $\therefore\ 2x(7x-8) - 7(7x+6) = 14x(x-5) - 4(4x+9).$

(Complete this solution and check it.)

NOTE 1. — Observe Steps 1 and 2 of the solution. Use similar steps in the following examples.

NOTE 2. — The result should be $\frac{2}{7}$. While you can check the solution by substituting $\frac{2}{7}$, possibly your teacher will permit you to check such fractional results by going over the solution most carefully.

18. $\frac{10}{m} - \frac{8}{m} = 1$

19. $\frac{3}{4m} + \frac{6}{5m} = \frac{13}{20}$

20. $\frac{5}{2x} - \frac{3}{x} + \frac{1}{12} = 0$

21. $\frac{8}{t} = \frac{4}{5t} - \frac{9}{5}$

22. $\frac{3}{y} - \frac{2}{5y} = 13$

23. $\frac{3-2r}{r} + \frac{6r+1}{3r} = \frac{20}{3}$

24. $\frac{5s+6}{2s} - \frac{7s-1}{5s} = -\frac{1}{2}$

25. $\frac{28}{3c} - \frac{14-2c}{5c} = \frac{4}{3}$

26. $\frac{6a-5}{2} = \frac{9a+5}{3} - \frac{25}{18a}$

27. $\frac{z-8}{2} + \frac{3z-6}{3z} = 2\tfrac{1}{2}$

28. $\frac{2}{x-2} - \frac{5}{x+2} = \frac{2}{x^2-4}$

Solution. — 1. The L. C. M. of the denominators is $x^2 - 4$. Multiply both members by $x^2 - 4$.

2. $$\overset{x+2}{(\cancel{x^2-4})} \cdot \frac{2}{(\cancel{x-2})} - \overset{(x-2)}{(\cancel{x^2-4})} \cdot \frac{5}{(\cancel{x+2})} = \overset{1}{(\cancel{x^2-4})} \cdot \frac{2}{(\cancel{x^2-4})}.$$

3. $$\therefore\ 2(x+2) - 5(x-2) = 2.$$

(Complete and check this solution.)

NOTE 1. — As soon as possible, do Step 2 mentally. Always write Step 3, however, carefully inclosing the numerators in parentheses as is done in this solution.

NOTE 2. — **When solving fractional equations, the only root obtained may be a number which makes one or more denominators zero.** Thus, in this example, if the root obtained is 2, the denominators of the first and third fractions become zero, and then these fractions do not have any meaning. (See § 68.) **In such cases this root is only an apparent root and the "equation" is only apparently an equation,** for a conditional equation is an equation *only if there is a number which satisfies it.* (See § 77.)

29. $\frac{15}{x-3} = 5$

30. $\frac{4}{m-1} = \frac{3}{m-2}$

31. $\frac{x-5}{x+2} = \frac{x+2}{x-5}$

32. $\frac{c+7}{c-5} - \frac{15}{c-5} = 2$

33. $\frac{x-5}{x+5}=\frac{x-2}{x+2}$

34. $\frac{4x}{2x-3}=\frac{3x}{2x-3}-\frac{1}{10}$

35. $\frac{12x^2+3x-1}{3x^2+2x-4}=4$

36. $\frac{1}{2x-4}=\frac{3}{10}-\frac{x-6}{x-2}$

37. $\frac{3}{x+3}=\frac{x+12}{x-3}-\frac{x}{x-3}$

38. $\frac{5x+1}{2x-3}-\frac{3}{5}=\frac{x}{2x-3}$

39. $\frac{9t}{t-2}=\frac{8t}{t-1}+1$

40. $\frac{a}{a+2}-\frac{a}{a-2}=\frac{a+20}{a^2-4}$

41. $\frac{2r}{r-4}+\frac{r}{r+4}=\frac{3r^2+16}{r^2-16}$

42. $\frac{2}{t+1}=\frac{6}{2t+1}-\frac{1}{t-3}$

43. $\frac{y^2+5y-30}{y^2-2y-15}+\frac{y}{y+3}=\frac{2y}{y-5}$

44. $\frac{2x^2+3x}{2x+1}+\frac{1}{3x}=x+1$

45. $\frac{x}{9}=\frac{x+1}{3}-\frac{2x^2-7}{9x-1}$

46. $\frac{2}{x}+\frac{x}{2}=-\frac{5}{2}$

(*Hint.* — Review § 105, page 147.)

47. $\frac{t}{3t+4}-\frac{3}{4t+1}$

48. $\frac{2a+3}{a+8}=\frac{2a+9}{3a+4}$

49. $\frac{m}{3m+4}-\frac{2}{7m-4}=0$

50. $4x-\frac{14-x}{x+1}=14$

51. $6r+\frac{35-3r}{r}=44$

52. $\frac{c}{c-1}-\frac{c-1}{c}=\frac{3}{2}$

53. $\frac{5x^2+6}{x^2-4}-\frac{3}{x-2}=\frac{5x+3}{x+2}$

54. $\frac{y}{5-y}-\frac{5-y}{y}=\frac{15}{4}$

EXERCISE 93

1. One sixth of a certain number is 10 less than one half the same number. What is the number?

2. There is a certain number such that the sum of one half of it, one third of it, and one fourth of it is 3 more than the number itself. What is the number?

3. Separate 62 into two parts such that five ninths of the greater shall be 7 more than one half of the smaller.

4. The sum of two numbers is 44. If the larger be divided by the smaller, the quotient equals $\frac{6}{5}$. What is the number?

5. One ninth of the result obtained by adding 12 to a certain number is one half the result obtained by subtracting 9 from the number. What is the number?

6. One sixth of "11 more than a certain number," added to one third of "10 less than the same number," gives 1. What is the number?

7. What number added to both terms of the fraction $\frac{5}{8}$ changes that fraction into one whose value is $\frac{3}{4}$?

8. What number added to the terms of the fractions $\frac{5}{8}$ and $\frac{2}{3}$ makes the resulting fractions equal?

9. A's age is twice B's age. If A's age 4 years from now be divided by B's age at that time, the quotient is $\frac{5}{3}$. What are their ages?

10. A and B are carrying loads of 40 pounds and 50 pounds respectively. How many pounds must be taken from A's load and added to B's so that A will have three fifths as much to carry as B?

11. The denominator of a certain fraction is 5 more than the numerator. If the denominator be increased by 1, and the numerator be decreased by 4, the value of the resulting fraction is $\frac{1}{3}$. What is the given fraction?

12. The numerator of a certain fraction is 9 less than twice the denominator. If 1 be added to both terms of the fraction, the resulting fraction has the value $\frac{3}{4}$. What is the fraction?

13. There are two consecutive even integers such that the larger is $\frac{4}{3}$ of the result obtained by subtracting 3 from the smaller. What are they?

14. A's age is two thirds of B's age. Ten years ago it was one half of B's age then. What are their ages now?

15. Three fifths of a certain angle equals three fourths of the complement of the angle. (See Ex. 4, page 23.) What is the angle?

16. There are three consecutive even integers such that one third of the first, plus one half of the second, plus one fourth of the third equals 15. What are they?

17. The perimeter of a certain triangle is 49 inches. The second side is $\frac{5}{4}$ of "1 less than the first side"; the third side is $\frac{2}{3}$ of "1 more than the first side." What are the sides of the triangle?

18. There are three consecutive integers such that one fifth of the largest is one fourth of the result obtained by subtracting 1 from the smallest. What are they?

19. Separate 58 into two parts such that the sum of one third of the greater and four sevenths of the smaller is 26.

20. The denominator of a certain fraction exceeds its numerator by 22. If 7 be subtracted from both terms of the fractions, the resulting fraction has the value $\frac{4}{15}$. What is the fraction?

21. One tenth of the supplement of a certain angle exceeds one half the complement of the angle by 1 degree. Find the angle.

22. One fifth of the supplement of a certain angle exceeds four sevenths of the complement of the angle by 5 degrees. Find the angle.

130. Work problems. If a man can do a piece of work in 8 days, then, in one day, he can do *one eighth* of it, and in 3 days he can do *three eighths* of it.

If a man can do the piece of work in x days, then, in one day, he can do $\frac{1}{x}$ *part* of it, and in 5 days he can do $5 \cdot \frac{1}{x}$ or $\frac{5}{x}$ part of it.

EXERCISE 94

1. If a man can plow a field in 12 days, what part can he plow in 1 day? in 4 days? in n days?

2. If a man can do a piece of work in y days, what part can he do in 1 day? in 3 days?

3. A can do a piece of work in 6 days and B can do it in 8 days.

a. How much can A do in 1 day? in x days?
b. How much can B do in 1 day? in x days?
c. How much can they do together in 1 day? in x days?

4. A can do a piece of work in 8 hours and B can do it in 5 hours. How long will it take them to do it if they work together?

Solution. — 1. Let $x =$ the no. of hr. it will take them to do it together.

2. Since A does $\frac{1}{8}$ in 1 hr., he will do $\frac{x}{8}$ in x hr.

Since B does $\frac{1}{5}$ in 1 hr., he will do $\frac{x}{5}$ in x hr.

$\therefore$ together they will do $\frac{x}{8} + \frac{x}{5}$ of it in x hr.

3. They complete the work in x hr.; that is, they do $\frac{40}{40}$ or 1 times it in that time.

$$\therefore \frac{x}{8} + \frac{x}{5} = 1.$$

(Complete the solution.)

Note 1. — Observe especially Step 3.

NOTE 2. — The essential facts of the solution can be charted.

PERSON	TIME ALONE	AMOUNT IN 1 HR.	AMOUNT IN x HR.
A	8 hr.	$\frac{1}{8}$	$\frac{x}{8}$
B	5 hr.	$\frac{1}{5}$	$\frac{x}{5}$
Together			$\frac{40}{40}$, or 1

$$\therefore\ \frac{x}{8}+\frac{x}{5}=1.$$

5. How long will it take A and B to do a piece of work together, if A can do it alone in 10 days and B in 20 days?

6. How long will it take A and B to do a piece of work together, if A alone can do it in 12 days and B in 9 days?

7. How long will it take A, B, and C to do a piece of work together, if A can do it alone in 10 days, B in 12 days, and C in 15 days?

8. A man can plow a certain field with his team in 8 days. His son can do it with their tractor in 2 days. How long will it take them to do it together?

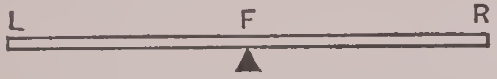
9. A can do a piece of work alone in 10 days and B in 15 days. After A had worked x days and B $(x+3)$ days, they found that they had done $\frac{3}{4}$ of the work. How long had each worked?

10. In a factory, there is one machine which can turn out a certain number of articles in 8 hours, a second which can do it in 16 hours, and a third in 20 hours. How long will it take to fill a rush order for this number of articles, if all three machines are worked at the same time?

11. A certain machine requires 25 hours to complete a certain job. After it had been running 17 hours, it became necessary to complete the work on a machine which requires 40 hours for the same job. How long should it have taken to complete the work?

131. Lever problems. A teeter board is one form of *lever.* The point which supports the lever is called the *fulcrum;* the parts of the lever to the right and left of the fulcrum are called the *lever arms.*

If the weights L and R just balance, it is well known that, if R moves farther to the right, while L is stationary, then the right side goes down; and if R moves toward the fulcrum, then the right side goes up. Thus, the influence of R upon the lever depends upon both the weight of R and its distance from the fulcrum.



The influence of a weight upon a lever is called its *leverage.* It can be shown that the *leverage of a weight is measured by the product of the weight and its distance from the fulcrum.*

Thus, if R weighs 50 pounds and is 4 feet from the fulcrum, its leverage is 200; and, if L weighs 80 pounds and is $2\frac{1}{2}$ feet from the fulcrum, its leverage is $80 \times 2\frac{1}{2}$ or 200 also. The two will balance.

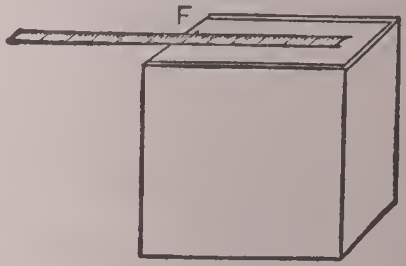
This truth may be tested in the following manner.

1. Remove a side and an end from a crayon box. Balance a stiff ruler on the edge of the box (the fulcrum).

2. Place two pennies 6 inches from the fulcrum on the left side. Find where four pennies must be placed on the right side of the fulcrum to balance them. (It should be 3 inches to the right.) Notice that $3 \times 4 = 12$ and that $6 \times 2 = 12$; that the leverages are equal.

3. Find where 3 pennies must be placed, on the right, to balance the 2 pennies on the left. Find the leverage of the 3 pennies and compare it with the leverages in Step 2.

4. Keeping the two pennies 6 inches to the left, place two others 2 inches to the right, and 2 four inches to the right. The ruler should be in perfect balance again.

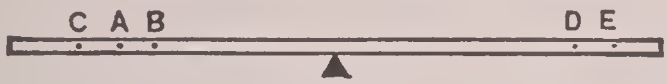


The leverage of the first two pennies on the right is 2×2 or 4; of the second two, is 2×4 or 8; $4 + 8 = 12$.

Example. — Suppose that the weights and distances in the figure on page 195 are:

A. 40 pounds; distance 5 feet. *B*. 45 pounds; distance 4 feet. *C*. 55 pounds; distance 6 feet. *D*. 60 pounds; distance 6 feet.

Where must *E* be placed so that the lever will balance?



Solution. — 1. The leverages are:

A. $5 \times 40 = 200.$ *C*. $6 \times 55 = 330.$ *E*. $50 \times x = 50\,x.$

B. $4 \times 45 = 180.$ *D*. $6 \times 60 = 360.$

2. $\therefore\ 200 + 180 + 330 = 360 + 50\,x$

$\therefore\ 350 = 50\,x,$ or $7 = x.$

That is, *E* must be 7 feet from the fulcrum.

Rule. — To balance a simple lever:

The sum of the leverages of all weights (forces) on one side of the fulcrum must equal the sum of the leverages of all weights (forces) on the other side of the fulcrum.

EXERCISE 95

1. A boy weighing 85 pounds sits 7 feet from the fulcrum and balances a boy who is sitting 6 feet from the fulcrum on the other side. What is the weight of the other boy?

2. C, weighing 105 pounds, sits 5 feet from the fulcrum on one side. Where must B, who weighs 75 pounds, sit, to balance him?

3. A, B, and C weigh $67\frac{1}{2}$, 70, and 90 pounds respectively. If A sits 6 feet from the fulcrum, and B sits on the same side, $4\frac{1}{2}$ feet from the fulcrum, where must C sit to balance them?

4. A man is carrying two packages, by balancing them over his shoulder at the ends of a stiff rod, 4 feet long. The packages weigh 25 and 35 pounds respectively. How far from his shoulder should the 25-pound package be carried?

5. Two girls, weighing 75 pounds and 85 pounds respectively, wish to sit at the ends of a teeter board, 10 feet long. How far from the lighter girl should the fulcrum be placed?

132. Additional rate, time, and distance problems.

EXERCISE 96

1. After reviewing § 85, page 111, give a simple arithmetical example involving rate, time, and distance.

2. What are the rule and the formula for finding the:

a. distance, when the rate and time are known?

b. rate, when the distance and time are known?

c. time, when the distance and rate are known?

3. The rate of one train is $\frac{3}{2}$ that of another. Let r represent the rate of the slower train.

a. Represent the rate of the faster train.

b. Represent the time each requires to go 125 miles.

4. Repeat Example 3 if the rate of the faster train exceeds that of the other train by 10 miles per hour.

5. The rate of a passenger train is twice that of a freight train. It goes 200 miles in $2\frac{1}{2}$ hours less time than the freight train requires for a trip of 150 miles. Find the rate of each.

Solution. — While the "times" are unknown, the equation is clearly to be obtained from the relation between the time each requires for its trip. These "times" must be expressed.

Let r = the rate of the freight train.

TRAIN	RATE	DISTANCE	TIME
Freight	r m. per hr.	150 m.	$\frac{150}{r}$ hr.
Passenger	$2r$ m. per hr.	200 m.	$\frac{200}{2r}$ hr.

$$\therefore \frac{200}{2r} = \frac{150}{r} - 2\tfrac{1}{2}.$$

(Complete and check this solution.)

6. Work Example 5 again, letting t = the number of hours the passenger train requires for the trip. Represent the "time" of the freight train; obtain the expressions for their rates; form and solve the equation.

7. The rate of a certain passenger train is twice that of a freight train. The passenger train travels 140 miles in 2 hours less time than the freight train requires for a trip of 105 miles. What are their rates?

8. The rate of a certain passenger train is 10 miles more per hour than the rate of an automobile. It takes the train $\frac{5}{7}$ as much time for a trip of 140 miles as it does the automobile. What is the rate of each?

9. The rate of one automobile party exceeds that of another party by 7 miles per hour. It goes 176 miles while the slower party is going 120 miles. What are their rates?

10. The rate of a certain freight train is $\frac{3}{4}$ that of a passenger train. It takes the freight train $1\frac{1}{2}$ hours longer than it does the passenger train to go 180 miles. What are the rates of the trains?

11. The rate of a certain passenger train exceeds that of a freight train by 10 miles per hour. Its time for a trip of 210 miles is $\frac{3}{2}$ that of the freight train for a trip of 100 miles. What are their rates?

River Problems

On a river, the direction in which the water is flowing is called *downstream*, and the opposite direction is called *upstream*. When going downstream, a boat is carried along by the *current* of the river and whatever force is exerted within the boat; when coming upstream, its progress is retarded by the current of the river.

12. The rate of the current of a river is 3 miles per hour.

a. What will be the rate downstream of some boys who are rowing at the rate of 5 miles per hour in still water? *b.* Upstream?

13. Repeat Example 12 if the boys are rowing at the rate of 3 miles per hour in still water.

14. How long will it take the boys in Example 12 to go:

a. 16 miles downstream? *b.* 16 miles upstream?

c. The round trip?

15. *a.* Express the time for the boys of Example 12 to go d miles downstream. *b.* d miles upstream.

c. Write the equation expressing the fact that the total time for the trip is 5 hours. Solve the equation.

16. Some boys who can travel 10 miles per hour in their motor boat want to make a trip on a river whose current is 3 miles per hour. How far may they go if they have only 7 hours for the round trip?

17. The rate of the current of a certain river is known to be 3 miles per hour. Some boys in a motor boat found that it took them as long to go 8 miles downstream as it did to go 3 miles upstream. What was their rate in still water?

SUPPLEMENTARY TOPICS

133. Literal equations.

Example. — Solve the equation $ax - a^2 = -b(b + x)$ for x.

Solution. — 1. $ax - a^2 = -b(b + x)$.

2. $\therefore ax - a^2 = -b^2 - bx$.

3. $\therefore ax + bx = a^2 - b^2$.

4. $\therefore x(a + b) = (a + b)(a - b)$.

5. D_{a+b} $\therefore x = \dfrac{(a + b)(a - b)}{(a + b)}$, or $a - b$.

Check this solution by substituting $(a - b)$ for x in the original equation.

NOTE. — This solution means that x is equal to $a - b$ regardless of the values of a and b.

EXERCISE 97

1. $7x + 10a = 16x - 17a$
2. $3(x - 2b) = 8x - 11b$
3. $a(bx - a) = b(a - bx)$
4. $rs(1 - 2x) = 2s^2x - r^2$
5. $a(a - x) = b(b + x)$
6. $a(x - a) = b(x - b)$
7. $c(x - c + 2d) = d(x + d)$
8. $r(x - s - r) = s(x - 2s)$
9. $mx - 2m^2 = mn - nx - n^2$
10. $m^2(x - 1) = mx - 1$
11. $2ax + 6b^2 = 3bx + 4ab$
12. $2ax + d = 3c - bx$
13. $a^2(x - 1) = b(b - 2a + ax)$
14. $ab(3x + 4) = 2(a^2 + 3b^2x)$
15. $b^2 = a(a + b^2x - a^2x)$
16. $c(x - 2) = d(x - 1)$
17. $a(x - a) = b(b - x)$
18. $\dfrac{m}{x} = 1 + \dfrac{m}{2}$
19. $\dfrac{12a}{x} = \dfrac{2x + 8a}{5x} - 3$
20. $\dfrac{a}{m} = \dfrac{1}{x} + \dfrac{2}{m}$
21. $\dfrac{x - 3m}{2x} = \dfrac{2}{3} - \dfrac{2x - 7m}{2x}$
22. $\dfrac{3x - 5a}{4} - \dfrac{9x - 7a}{12} = \dfrac{2}{3x}$
23. $\dfrac{x}{2} = \dfrac{1 + 2ax}{2a} - \dfrac{2x + 1}{a^2}$
24. $\dfrac{4a}{x - 2a} = 1$
25. $\dfrac{6}{x - 3c} = \dfrac{5}{x - 5c}$
26. $\dfrac{a + b}{2} = \dfrac{c}{x - 2}$
27. $\dfrac{x + a}{x - 3a} + \dfrac{4a}{x - 3a} = 5$
28. $\dfrac{3x}{2x + 3c} - \dfrac{2x}{2x - 3c} = \dfrac{2x^2 - 15c^2}{4x^2 - 9c^2}$
29. $\dfrac{6x + 5a}{2x^2 - 2ax} + \dfrac{2a}{x^2 - a^2} = \dfrac{3x}{x^2 - a^2}$
30. $\dfrac{9}{3x - 5a} - \dfrac{2}{a - 2a} = \dfrac{1}{x - 3a}$

134. Deriving and using formulæ.

Example. — Derive a formula for the *rate* in terms of the *amount*, the *principal*, and the *time* in simple interest problems.

Solution. — 1. The formula for the amount is

$$A = P + \frac{Prt}{100}.$$

2. M_{100} $\quad 100\,A = 100\,P + Prt.$

3. S_{100} $\quad 100\,A - 100\,P = Prt.$

4. D_{Pt} $\quad \dfrac{100\,A - 100\,P}{Pt} = r$

5. $\quad$ or $\quad r = \dfrac{100(A - P)}{Pt}.$

This formula enables us to find r when A, P, and t are known. Thus, if a man receives \$3500 at the end of 6 years from an investment of \$2400, what rate of simple interest has his money earned?

Here, $\quad A = \$3500,\ P = \$2400,\text{ and } t = 6.$

$$\therefore\ r = \frac{\cancel{100}(3500 - 2400)}{\underset{24}{\cancel{2400}} \times 6} = \frac{1100}{144} = 7.6;$$

that is, the rate is 7.6%.

In the following list of examples, a number of formulæ, taken from physics, chemistry, geometry, and engineering, are given. It is impossible to show in this text how these formulæ are derived, as that calls for special knowledge of these various subjects.

EXERCISE 98

Solve each of them for the letters indicated:

1. $A = ab$. *a.* Solve for a; *b.* for b.

2. $A = \dfrac{ab}{2}$. *a.* Solve for a; *b.* for b.

3. $C = 2\,\pi r$. Solve for r.

4. $V = lwh$. *a.* Solve for w; *b.* for h.

5. $V = \frac{1}{3}\,bh$. *a.* Solve for b; *b.* for h.

6. $F = \frac{9}{5}\,C + 32$. Solve for C.

7. $A = \dfrac{a(b+c)}{2}$ is a formula from geometry.

a. Find A when $a = 15$; $b = 24$; and $c = 20$.
b. Find c when $A = 550$; $b = 30$; and $a = 22$.
c. Solve the formula for a; *d.* for b.

8. $A = P + \dfrac{Prt}{100}$.

a. Solve for P. *b.* for t.

9. $T = \dfrac{1}{a} + t$ is a formula from physics.

a. Solve it for t. *b.* for a.

10. $mg - T = mf$ is a formula from physics.

a. Solve it for T; *b.* for f; *c.* for m.

11. $u = \left(\dfrac{m}{M+m}\right)v$ is a formula from physics.

a. Solve it for m; *b.* for M.

12. $C = \dfrac{Kab}{b-a}$ is a formula from physics.

a. Solve it for a; *b.* for b.

13. $s = \dfrac{c-b}{b-a}$ is a formula from physics.

a. Solve it for b; *b.* for a.

14. $h = k(1 + \frac{1}{273}t)$ is a formula from physics.

a. Solve it for t; *b.* for k.

15. $l = \dfrac{L-M}{Mt}$ is a formula from physics.

a. Solve it for L; *b.* for M.

16. $w = \dfrac{5a}{1 + \dfrac{l^2}{600d^2}}$ is a formula from engineering.

a. Find w, correct to two decimals, when $a = 18$, $l = 12$, and $d = 10$.

b. Solve the formula for a.

17. $p = \dfrac{ad^2}{t} + d$ is a formula from engineering.

a. Find p when $a = .56$, $d = \frac{1}{4}$, $t = \frac{3}{8}$.

b. Solve the formula for t.

18. $C = \dfrac{E}{R + \dfrac{r}{n}}$ is a formula from physics.

a. Simplify it in the right member.

b. Solve it for n; *c.* for r.

19. $F = \dfrac{mv^2}{gr}$ is a formula from physics.

a. Find F when $m = 150$, $v = 25$, $g = 32$, and $r = 5$.

b. Solve it for r; *c.* for m.

20. $\dfrac{1}{f} = \dfrac{1}{p} + \dfrac{1}{q}$ is a formula from physics.

a. Find p when $f = 30$ and $q = 40$.

b. Solve it for p; *c.* for q.

135. Equations involving decimals.

Example. — Solve the equation $.7x - 1 = .36x - .15$.

Solution a. — 1. $.7x - 1 = .36x - .15$.
2. M_{100} $70x - 100 = 36x - 15$.

This solution may be completed in the usual way, and checked by substitution.

Solution b. — 1. $.7x - 1 = .36x - .15$.
2. $\therefore .7x - .36x = 1 - .15$.
3. $\therefore .34x = .85$.
4. D_{34} $x = 2.5$.

$$\begin{array}{r} 2.5 \\ {}^{\times}34\overline{){}^{\times}85.00} \\ \underline{68} \\ 17\,0 \\ \underline{17\,0} \end{array}$$

EXERCISE 99

Solve the following equations:

1. $.5x + .5 = 6.1 - .3x$
2. $.9m - .75 = .3m - 3.75$
3. $.13 + .04p = .11p - .22$
4. $.05a - .12 = .16 - .09a$
5. $2.1 - 1.5z = -.8z - .7$
6. $3 + 1.7c = 2.7c + 2.2$
7. $.3(8 - 5a) = .2(a - 5)$
8. $.3(x - 2) = 5(.2x - 1.1)$

9. $.5 = .3(x - 2) - (x - 6)$

10. $6(.2 - .3x) + .3 = .3(4x - 5)$

11. $.2x - .03x - .113x = .01 + .161$

12. $.3x - .02 - .003x = .7 - .06x - .006$

13. $.3(1.2x - 5) = 14 + .05x$

14. $.7(x + .13) = .03(4x - .1) + .5$

15. $3.3x - \dfrac{.72x - .55}{.5} = .1x + 9.9$

136. Meaning and use of ratio. The ratio of one number to another is the quotient of the first divided by the second.

Thus, the ratio of 2 to 6 is $\frac{2}{6}$ or $\frac{1}{3}$. It means that 2 is $\frac{1}{3}$ of 6.

The ratio of a to b is the fraction $\dfrac{a}{b}$.

The first number (the **antecedent**) is made the numerator of the fraction, and the second (the **consequent**) the denominator.

The ratio of two concrete quantities has meaning only if they are of the same kind. Their ratio is the quotient of their measures in terms of the same unit of measure.

Thus, the ratio of a segment 2 yd. long to a rectangle having 3 sq. yd. of surface is without meaning.

The ratio of a line 2 yd. long to one 10 ft. long is 6 ft. ÷ 10 ft., or $\frac{3}{5}$.

EXERCISE 100

What is the ratio of:

1. 2 to 12?

2. 3 to 15?

3. 1 pt. to 2 qt.?

4. 3 oz. to 1 lb.?

5. 10 rd. to 1 mi.?

6. 3 d. to 2 wk.?

7. 6 in. to 3 ft.?

8. 1500 lb. to 1 t.?

9. $2\frac{1}{2}$ to 5?

10. $3\frac{1}{3}$ to $4\frac{1}{2}$?

11. Out of an income of \$2500, a man spends \$400 for rent. What is the ratio of his rent to his income?

12. *a.* The altitude of rectangle R is 8 in. long and its base is 24 in. What is the ratio of its altitude to its base?

b. The altitude of rectangle S is 9 in., and its base is 36 in. What is the ratio of its altitude to its base?

c. How do the ratios compare?

13. *a.* The altitude and base of one rectangle are 8 ft. and 12 ft. respectively. What is their ratio?

b. The altitude and base of another rectangle are 11 ft. and 33 ft. respectively. What is their ratio?

c. How do the ratios compare?

14. *a.* A purchased 12 t. of coal and B, 15 t. What was the ratio of the amounts purchased?

b. A paid $108 for his coal and B, $135. What was the ratio of their expenditures?

c. How do the ratios compare?

137. Meaning and use of proportion. If the ratio of two numbers equals the ratio of two other numbers, the four numbers form a **proportion**.

Thus, $\frac{2}{4} = \frac{6}{12}$. Therefore 2, 4, 6, and 12 form a proportion.

As a fractional equation, we read it *two fourths equals six twelfths.*

As a proportion, we read it, 2 is to 4 as 6 is to 12; meaning 2 has the same relation to 4 as 6 has to 12.

Similarly, the numbers a, b, c, and d are proportional if $\frac{a}{b} = \frac{c}{d}$.

Read it, *a is to b as c is to d.*

a, b, c, and d are called the **first, second, third,** and **fourth terms of the proportion.**

They are placed in the proportion in that order. a and d are the **extremes** of the proportion; b and c are the **means** of the proportion.

A proportion is really only a very simple fractional equation and does not present any difficulties.

Example. — Find x from the proportion $\frac{m}{np} = \frac{c}{nx}$.

Solution. — 1. $$\frac{m}{np} = \frac{c}{nx}.$$

2. M_{npx} $$\overset{x}{\cancel{npx}}\cdot\frac{m}{\cancel{np}} = \overset{p}{\cancel{npx}}\cdot\frac{c}{\cancel{nx}}.$$

3. $$\therefore\ mx = cp.$$

4. D_m $$x = \frac{cp}{m}.$$

EXERCISE 101

Find x in each of the following proportions:

1. $\frac{x}{4} = \frac{5}{3}$

2. $\frac{2}{x} = \frac{3}{5}$

3. $\frac{7}{16} = \frac{x}{5}$

4. $\frac{9}{24} = \frac{3}{x}$

5. $\frac{x-1}{2} = \frac{5}{3}$

6. $\frac{t-3}{t+5} = \frac{4}{6}$

7. $\frac{x}{a} = \frac{b}{c}$

8. $\frac{m}{x} = \frac{n}{r}$

9. $\frac{p}{q} = \frac{r}{x}$

10. $\frac{r^2}{sx} = \frac{r}{t}$

11. $\frac{ac}{x} = \frac{c^2}{d}$

12. $\frac{x-a}{b} = \frac{c}{1}$

13. $\frac{x}{a-x} = \frac{2}{3}$

14. $\frac{x}{a-x} = \frac{m}{n}$

15. If it takes 75 gal. of road oil to cover 150 sq. yd., how many gallons will it take for 1525 sq. yd.?

Solution. — 1. In this example, it is implied that the amount of oil for 1525 sq. yd. is to the amount required for 150 sq. yd. as 1525 sq. yd. is to 150 sq. yd.

NOTE. — Observe, am't of oil is to am't of oil as no. of sq. yd. is to no. of sq. yd.

2. Let n = no. of gal. of oil required for 1525 sq. yd.

3. $$\therefore\ \frac{n}{75} = \frac{1525}{150}.$$

(Complete and check the solution.)

16. If 20 acres of ground cost $3750, what should 35 acres cost at the same rate?

17. If 14 lb. of sugar can be bought for $1, what should 25 lb. cost at the same rate?

18. The weight of 18 cakes of ice was 5850 lb. What should be the weight of 30 cakes having the same average weight?

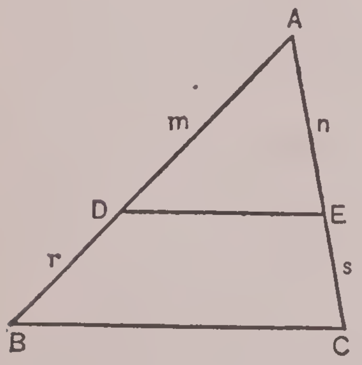
19. Two families together purchased 75 shrubs for $21.75. One family took 45 of them. How much should that family pay?

20. A train travels 130 miles in 4 hours. How many hours should it take a train to travel 250 miles at the same rate?

21. Separate a line 45 inches long into two parts which have the ratio 4 to 9. (Solve it in the two ways suggested below.)

Solution a. Let $x =$ shorter part; and $\therefore 45 - x =$ longer part.
Solution b. Let $4x =$ shorter part; and $\therefore 9x =$ longer part.

22. Separate a line 84 inches long into three parts proportional to 2, 3, and 4. (The method used in Solution *b* above should be used in this case.)



23. Separate 280 into four parts proportional to 2, 3, 4, and 5.

24. In a triangle in which DE is parallel to BC, $m : r = n : s$.

To test this truth: (*a*) measure m, n, r, and s; (*b*) find the value of the ratio $m : r$ and of $n : s$; (*c*) compare these two ratios.

This truth may be tested in any triangle. It may be expressed thus: *the upper segment on one side is to the lower segment on that side as the upper segment on the other is to the lower segment on the other.*

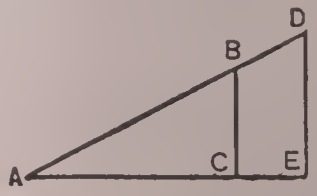
25. If $AD = 7$, $DB = 4$, $AE = 8$, find EC.

26. If $AB = 12$, $AD = 5$, $AC = 14$, find AE.

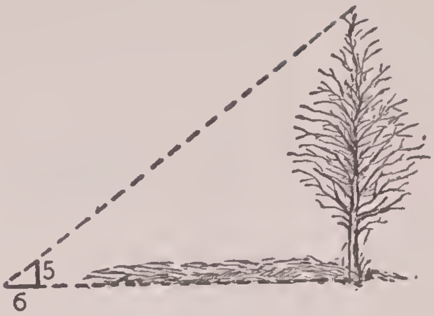
Hint. — Let $AE = x$, and $CE = 14 - x$.

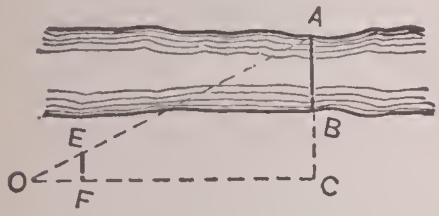
27. In the figure at the right, prove by measurement that

$$\frac{AC}{AE} = \frac{BC}{DE}.$$



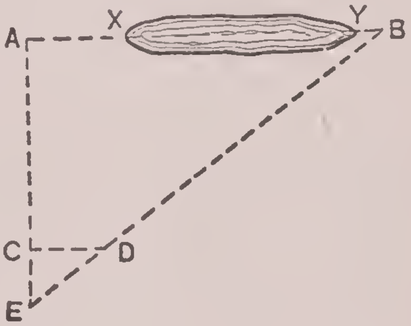
28. Using the fact proved in Example 27, find the height of the tree, using the facts given in the figure at the right.





29. Suppose that EF and AC are perpendicular to OC in the adjoining figure. Suppose that $EF = 10$ feet, $OF = 12$ feet, $OC = 150$ feet, and $BC = 20$ feet. Determine AB.

30. Suppose that CD and AB are perpendicular to AE in the adjoining figure; that $AX = 5$ feet, $YB = 8$ feet, $AE = 750$ feet, $CE = 25$ feet, and $CD = 30$ feet. Find XY.

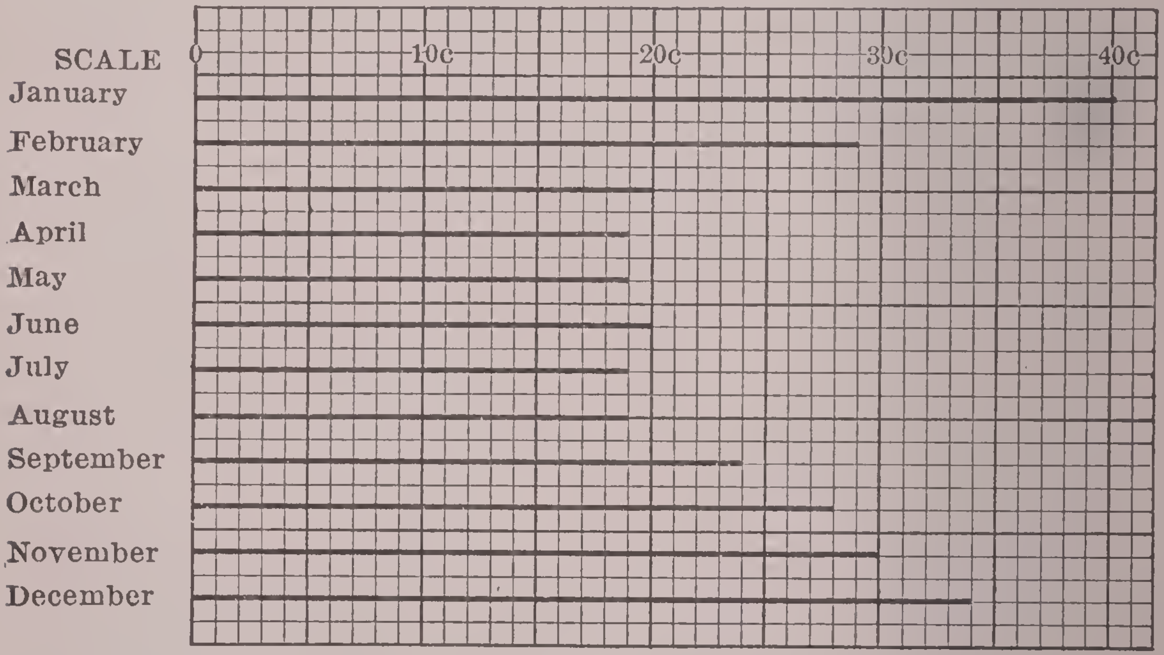


XI. GRAPHICAL REPRESENTATION

138. Graphical representation of statistics is illustrated in the graph below.

Explanation of the Graph. — Opposite the word **scale**, observe the points marked 0¢, 10¢, 20¢, etc. This scale means that a *segment* (line, you may call it) from 0 to 10 represents 10 cents; one from 0 to 30 represents 30 cents, etc.

Opposite each month is a segment whose length, *according to the scale*, represents the price of eggs that month. Thus, the price in January was 40 cents.



Note 1. — This graph is drawn on **coördinate paper**. You can make such graphs on ordinary paper by drawing parallel segments according to some scale. Thus, you can let $\frac{1}{8}$ in. represent 1¢; then, in this example, the January price will be represented by a segment 5 in. long, since $40 \cdot (\frac{1}{8} \text{ in.}) = 5 \text{ in.}$

Note 2. — The segments can be drawn vertically or horizontally.

Note 3. — Instead of drawing a segment, a narrow rectangle may be made.

Thus, opposite January,

The resulting graph is called a **bar graph**.

EXERCISE 102

1. In what months was the price lowest? During how many months did the price differ but little from this lowest price? What accounts for this low price during these months?

2. Determine from the graph the prices, and fill out the following table:

Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Price per doz.												

Represent graphically the statistics given in each of the following tables of statistics.

3. The wholesale price of eggs on the first Tuesday of each month of a certain year in Chicago.

Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Price per doz.	34¢	27¢	19¢	18¢	18¢	17¢	16¢	17¢	22¢	24¢	27¢	30¢

4. The percentage of pupils studying algebra in one school, receiving each of certain grades.

Grade	95–100	88–94	81–87	75–80	In-complete	Failed
Per cent receiving it	4	18	27	26	11	13

5. Tabular presentation of the average retail price of butter.

During the Year	1895	1900	1905	1910	1915	1920
Cost of 1 pound	25¢	26¢	29¢	36¢	36¢	67¢

6. Tabular presentation of the population of the United States.

Date	1860	1870	1880	1890	1900	1910	1920
Number of millions	30	39	50	63	76	92	106

7. Tabular presentation of the distance in which an auto can be stopped when going at certain speeds.

Speed in Miles per Hour	10 m.	15 m.	20 m.	25 m.	30 m.
Number of feet before it can be stopped	9 ft.	21 ft.	37 ft.	58 ft.	83 ft.

8. The average temperature each month at Chicago.

Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Average temperature	24°	25°	34°	46°	56°	66°	72°	71°	65°	53°	39°	29°

139. Variables. In each of the examples of the previous section there was a table of values which some *changing number* had under *certain changing conditions.*

Thus, in Example 3, page 209, the "changing number" was the *price* of eggs and the "changing conditions" was the *month of the year.*

A changing number is called a variable. The variable is said to **vary with** or **vary according to** the changing conditions.

The relation between the variable and the changing condition which caused the variation **was expressed**

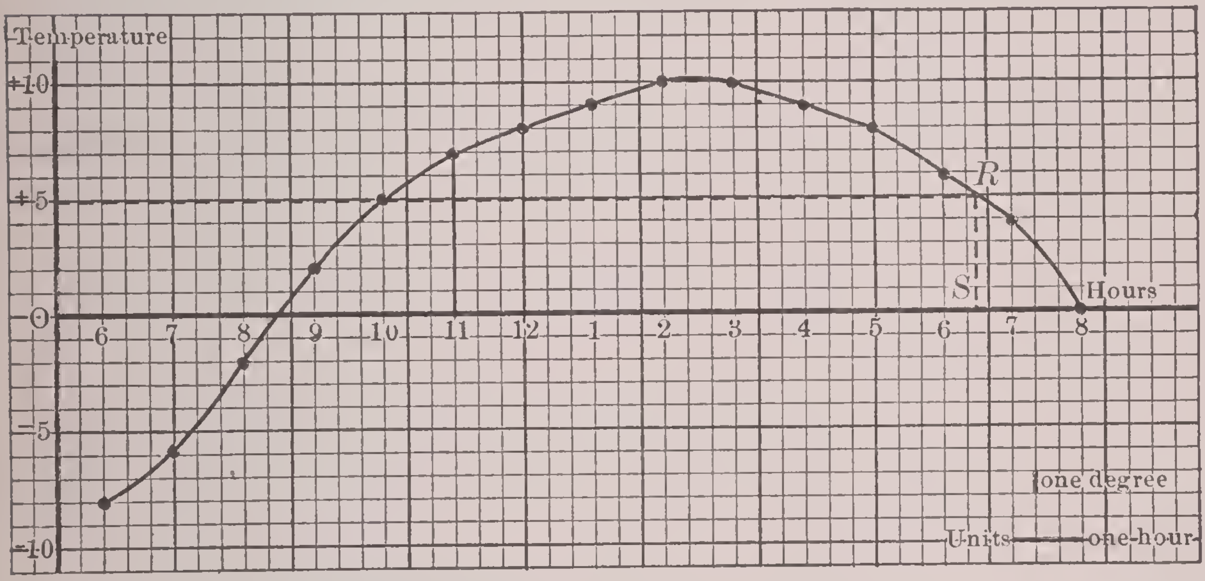
a. **in tabular form,** in the statement of the example;

b. **in graphical form,** when you solved the example.

140. Graphs involving positive and negative numbers.

Example. — Draw the graph representing the following temperature readings.

6 A.M.	− 8°	11 A.M.	+ 7°	4 P.M.	+ 9°
7 A.M.	− 6°	12 M.	+ 8°	5 P.M.	+ 8°
8 A.M.	− 2°	1 P.M.	+ 9°	6 P.M.	+ 6°
9 A.M.	+ 2°	2 P.M.	+ 10°	7 P.M.	+ 4°
10 A.M.	+ 5°	3 P.M.	+ 10°	8 P.M.	0



Explanation of this Graph. — On the horizontal line, called **the horizontal axis**, points representing 6 A.M., 7 A.M., etc., are placed at equal distances.

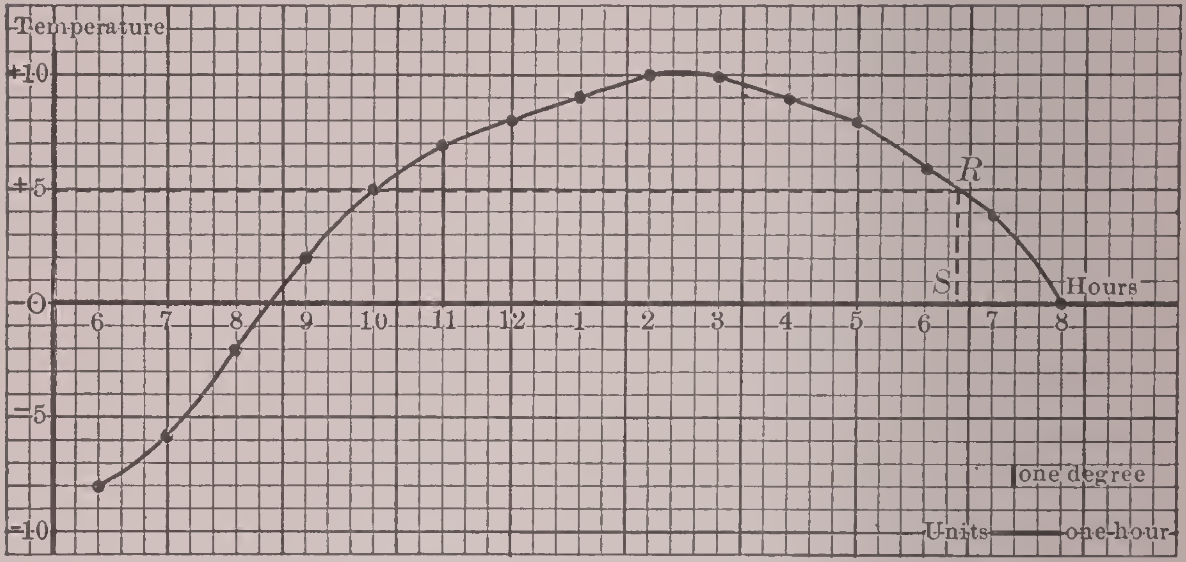
On the heavy vertical line, called **the vertical axis**, + 5 means + 5°; − 10, means − 10°. Each space vertically represents 1°. This is called the **unit. Segments drawn upward** from the horizontal axis **represent positive temperature; segments drawn downward** from the horizontal axis **represent negative temperature.**

Thus, the black vertical line above the point representing 11 A.M. is 7 units long. The temperature, then, was + 7°.

The curved line, **which is the graph proper**, was obtained as follows. Instead of drawing lines at each hour like the one at 11 o'clock, the points were marked where these lines would end. (See the figure.) **These points were connected by the smoothest possible curved line.**

This graph pictures the change in temperature from hour to hour during the day.

141. A graph may enable us to arrive at new information connected with the statistics being graphed.



Example 1. — Referring to the figure above, what was the temperature at 6 : 30 P.M.?

Solution. — 1. Point S on the horizontal line corresponds to 6 : 30 P.M.

2. Go from S vertically to R on the graph. RS represents the temperature at 6 : 30 P.M. Go from R horizontally to + 5 on the vertical axis.

3. ∴ the approximate temperature was + 5°.

Example 2. — When was the temperature + 4°?

Solution. — 1. Without making any marks on the graph, start at the point on the vertical axis corresponding to + 4°.

2. Go horizontally to the curved line, and from there go down to the horizontal axis. You arrive at a point two spaces to the right of 9; this point marks 9 : 40 A.M.

3. ∴ it was + 4° at 9 : 40 A.M.

4. Also, a line from + 4° drawn horizontally crosses the graph at a second point which is above the point corresponding to 7 P.M. Hence, it was + 4° also at 7 P.M.

EXERCISE 103

1. Referring to the graph above, what was the approximate temperature at 6 : 30 A.M.? at 8 : 30 A.M.? at 3 : 40 P.M.?

2. Draw the graph representing the following table:

6 A.M.	+ 5°	9 A.M.	+ 15°	12 M.	+ 20°	3 P.M.	+ 4°
7 A.M.	+ 8°	10 A.M.	+ 17°	1 P.M.	+ 18°	4 P.M.	− 2°
8 A.M.	+ 12°	11 A.M.	+ 19°	2 P.M.	+ 12°	5 P.M.	− 5°

3. If possible, get the temperature readings in your town for to-day, and represent them graphically.

4. *a.* Complete the following cost table at 32¢ per pound of articles weighing 0 lb. to 5 lb. inclusive.

WEIGHT	0	1 lb.	2 lb.	3 lb.	4 lb.	5 lb.
COST	0¢	32¢	64¢	?	?	?

b. Represent the above table graphically.

Hint. — Let 1 space horizontally represent $\frac{1}{8}$ lb. Let 5 spaces vertically represent 20¢.

c. From your graph find the cost of $2\frac{1}{4}$ lb.; $3\frac{1}{8}$ lb.; $1\frac{3}{8}$ lb.; $4\frac{5}{8}$ lb.

5. Repeat Example 4 if the cost is $12\frac{1}{2}$¢ per pound, for 0, 2, 4, 6, 8, and 10 pounds.

6. *a.* The area of the rectangle with altitude 5 and base b is given by the formula $A = 5b$.

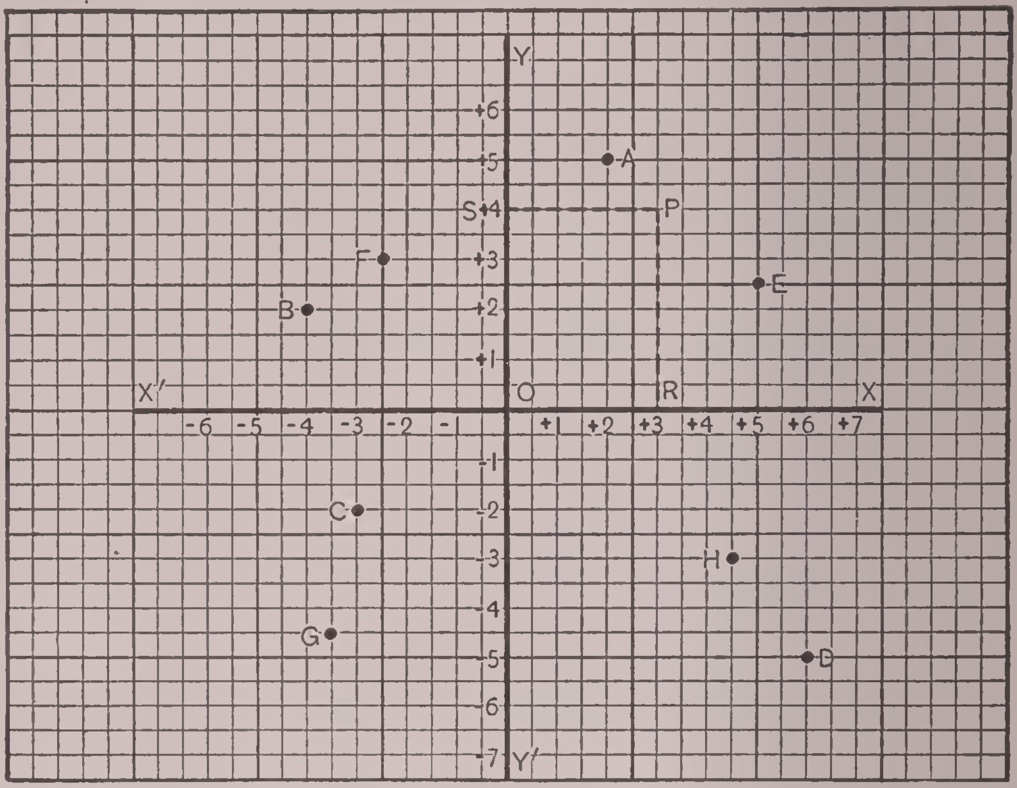
b. Make a table showing the area A, when $b = 0, 2, 4, 6, 8, 10$.

c. Represent this table graphically.

7. The compound interest, to the nearest cent, on $1 invested at 4% compounded quarterly is given below. Express this information graphically.

NO. OF YEARS	1	2	3	4	5	6	7	8	9	10
COMPOUND INTEREST	.04	.08	1.13	1.17	1.22	1.27	1.32	1.37	1.43	1.49

142. Definitions. In the figure below:



a. lines XX' (read ex ex prime) and YY', drawn at right angles, are called the **axes.**

b. XX' is called the **horizontal axis** or **X-axis.**

c. $Y\,Y'$ is called the **vertical axis** or **Y-axis.**

d. Point O, where the axes intersect, is called the **origin.**

e. PR, perpendicular from P to XX', is the **ordinate of** P.

f. PS, perpendicular from P to $Y\,Y'$, is the **abscissa of** P (plural, *abscissæ*).

g. The ordinate (PR) and abscissa (PS), together, are the **co-ordinates** of P.

Since $ORPS$ is a rectangle, $PR = OS$; *therefore the ordinate of P equals OS.* Similarly, *the abscissa of* $P = OR$.

Distances up from XX' **are positive; distances down from** XX' **are negative.** Notice on OY the **positive scale,** $+1$ $+2$, $+3$, etc., and on OY' the **negative scale,** -1, -2, -3, etc.

Distances to the right from YY' **are positive; distances to the left are negative.** Again notice the positive and negative scales on XX'.

Thus, the abscissa of P is $+3$; and the ordinate of P is $+4$. P is called the *point* $(+3, +4)$. This means that P is located 3 spaces to the right and 4 spaces up.

When giving the coördinates of a point, **the abscissa is always given first.**

NOTE. — The numbers of the scale should always be directly below the points of the horizontal axis, and exactly to the left of the points of the vertical exis.

The scale need not be the same on the two axes.

EXERCISE 104

1. What is the abscissa of A? F? G? H?

2. What is the ordinate of E? B? C? D?

3. What are the coördinates of A? E? B? G? H?

4. On a similar sheet of paper, or on the blackboard, locate the following points:

a. $(3, 5)$	*b.* $(-4, 5)$	*c.* $(-3, -6)$	*d.* $(+4, -3)$
e. $(0, 7)$	*f.* $(+5, 0)$	*g.* $(0, -10)$	*h.* $(-8, 0)$

5. *a.* Locate points $R: (2, 3)$ and $S: (-5, -3)$. Draw RS.

b. Locate points $T: (-3\frac{1}{2}, 1\frac{1}{2})$ and $W: (3, -1)$.

c. What are the coördinates of the point where RS intersects TW?

6. *a.* Locate $C: (5, 4)$; $D: (-5, +6)$; $E: (-7\frac{1}{2}, -\frac{1}{2})$; $F: (2\frac{1}{2}, -2\frac{1}{2})$.

b. Draw CD, DE, EF, and CF. What figure is formed?

c. Draw its diagonals. What are the coördinates of their point of intersection?

7. *a.* Locate $M: (+6, -4)$ and $N: (-6, +2)$, and draw MN.

b. What are the coördinates of the mid-point of MN?

143. Graphing the relation between two variables connected by a formula or equation.

Example. — Represent graphically the relation between x and y, when x and y satisfy the equation $2x + y = 4$.

Solution. — 1. Select values of x and determine the corresponding values of y.

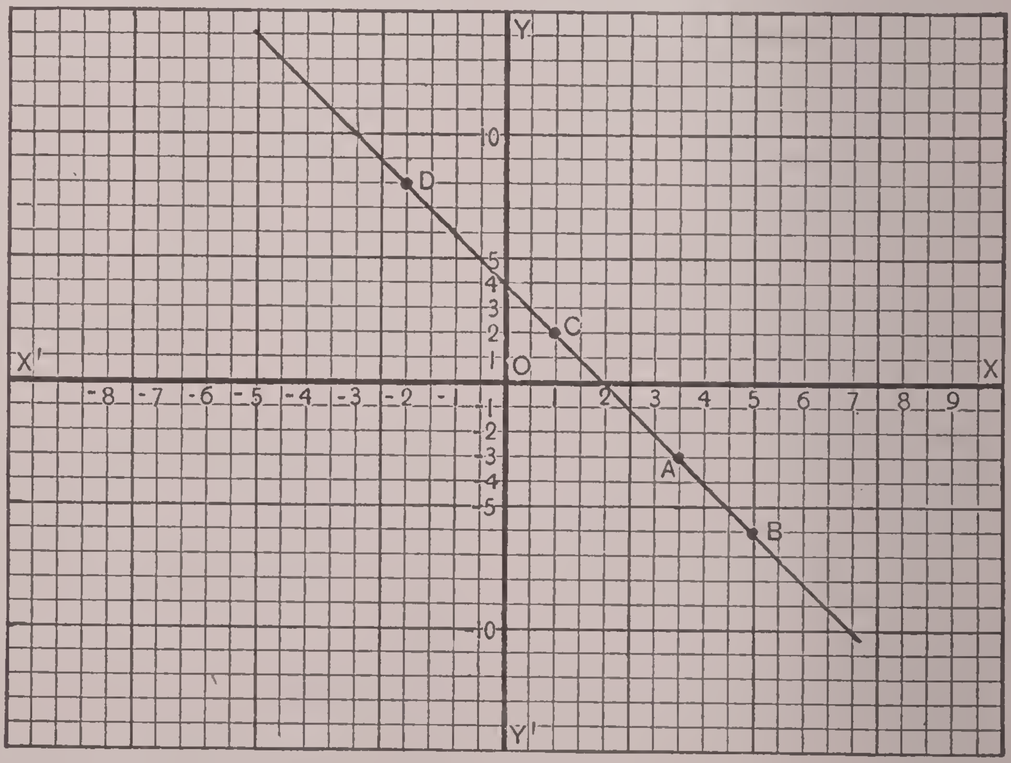
Thus, when $x = -5$, $2(-5) + y = 4$; $-10 + y = 4$; $y = 14$.
when $x = +6$, $2(6) + y = 4$; $12 + y = 4$; $y = -8$.
Similarly,

When $x =$	-5	-3	-1	0	$+2$	$+4$	$+6$	$+7$
Then $y =$	$+14$	$+10$	$+6$	$+4$	0	-4	-8	-10

2. Use each pair of values so obtained as coördinates of a point. Represent each point by a small dot.

Thus, place a dot at $(-5, +14)$; at $(-3, +10)$; etc.

3. Draw the smooth line connecting these points.



4. This is the required graph. Notice that it is a straight line.

Note 1. — The computation of Step 1 should be done mentally.

Note 2. — This graph is usually called the graph of the equation $2x + y = 4$.

EXERCISE 105

1. *a.* What are the coördinates of C? Do they satisfy the equation?

b. Similarly, test the coördinates of B and D.

2. *a.* Consider point (5, 5), not on the graph. Do these coördinates satisfy the equation?

b. Try the coördinates of any other point not on the graph.

3. *a.* Express graphically the relation $y = x - 3$.

Hint. — Let $x = -3, -2, -1, 0, 1, 2, 3, 4, 5$.

b. What does the graph appear to be?

c. Select a point not on this graph. Do its coördinates satisfy the given equation?

4. *a.* Express graphically the relation $x + 2y = 6$.

Hint. — Follow the steps of the illustrative example. Let $x = -4, -2, 0, 2, 4, 6, 8, 10$.

b. What does the graph appear to be?

c. Select some point on the graph. Do its coördinates satisfy the given equation?

d. Select a point not on the graph. Do its coördinates satisfy the equation?

5. Repeat Example 4 for the equation $x - 2y = 6$.

6. Draw the graph of the equation $x + y = 5$.

7. Draw the graph of the equation $x - y = 8$.

144. Summary. *a.* **The graph of an equation of the first degree having two variables is a straight line.**

For this reason these equations are called **linear equations.**

b. **The coördinates of every point on the line satisfy the equation.**

c. **The coördinates of every point not on the line do not satisfy the equation.**

145. Short method of drawing the graph of a linear equation having two variables.

Since the graph is a straight line, and since a straight line can be drawn with a ruler as soon as two of its points are known, then

Rule. — To draw the graph of a linear equation having two variables:

1. Select a value for one variable. Determine the corresponding value of the other variable. These are the coördinates of one point on the graph. Locate it.

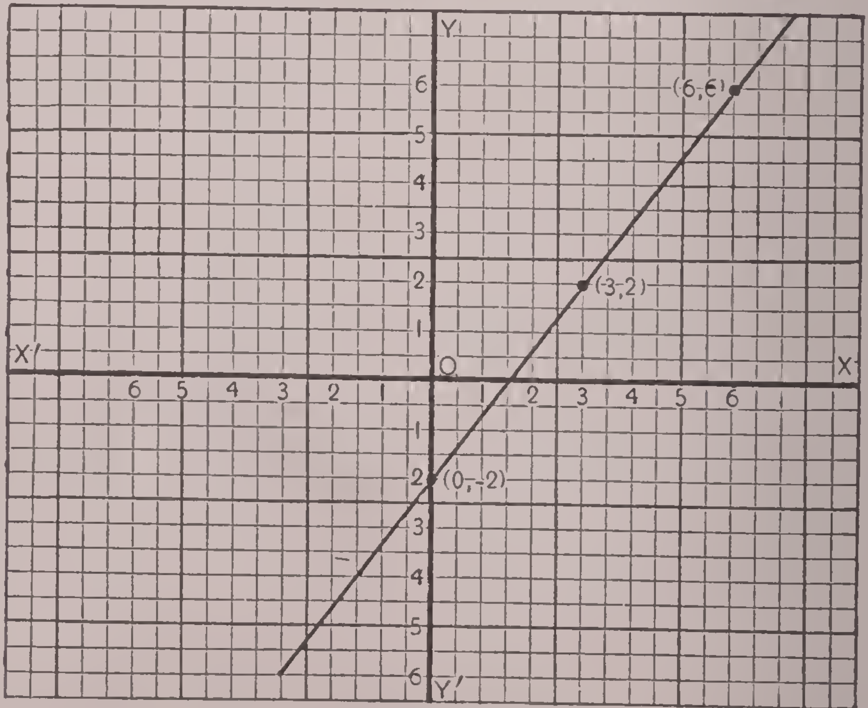
2. Locate a second point in the same manner.

3. Draw the straight line passing through these two points.

4. Find a third point as in Step 1. If it falls on the graph, the work has been done correctly.

Example. — Draw the graph of $4x - 3y = 6$.

Solution. — 1. Let $x = 0$. Then $y = -2$. $\left\{\begin{array}{l}\text{Obtained mentally.}\\ \text{See Note, below.}\end{array}\right.$
2. Let $x = 3$. Then $y = 2$.
3. Locate the points and draw the graph.



Check.—Let $x=6$. Then $y=6$. Does this point lie on the line? Yes.

NOTE. — The easiest value of x to use is $x = 0$, as in this example, for then $-3y = 6$ and $y = -2$.

In an equation like $3x - 5y = 11$, when $x = 0$, $-5y = 11$, and $y = -2\frac{1}{5}$. While this is correct, the point $(0, -2\frac{1}{5})$ cannot be located easily. In such cases, try $y = 0$. Then $3x = 11$, and $x = 3\frac{2}{3}$. This is equally unsatisfactory.

Try $x = 1$, then $3 - 5y = 11$; $-5y = 8$; $y = -1\frac{3}{5}$. Unsatisfactory.

Try $x = 2$, then $6 - 5y = 11$; $-5y = 5$; $y = -1$. This is a satisfactory point.

Try $x = -3$, then $-9 - 5y = 11$; $-5y = 20$; $y = -4$.

Do not give up until you get at least two satisfactory pairs of numbers which satisfy the equation. If necessary, use fractional values of the variables.

EXERCISE 106

Draw the graph of each of the following equations, each on a separate piece of coördinate paper.

1. $x - y = 5$

2. $2x - y = 3$

3. $3x + y = 4$

4. $\frac{1}{2}x - y = 5$

5. $2y = 6 - x$

6. $y - 3x = 0$

7. $4x + y = 0$

8. $4x + 5y = 20$

9. $3x - 2 = y$

10. $3x + 5y - 15 = 0$

Supplementary Examples

11. $y = x + 2.5$

12. $y = 3x - .75$

13. $y = .5x + .8$

14. $y = 5.3 - .7x$

15. $\frac{1}{2}x - \frac{1}{3}y = 5$

(*Hint.* — M_6.)

16. The sum of two numbers is 9. Letting x and y represent these numbers, form the equation expressing the fact that their sum is 9. Then draw the graph of the equation.

17. One number exceeds twice another number by 5. Express these numbers by x and y; form the equation; draw its graph.

146. Two linear equations having two variables.

A solution of an equation having two (or more) variables consists of a value of each which, together, satisfy the equation.

Thus, $x = 1$, $y = 4$ is a solution of $x + y = 5$, since $1 + 4 = 5$. Also, $x = -8$, $y = 13$ is a solution since $-8 + 13 = 5$.

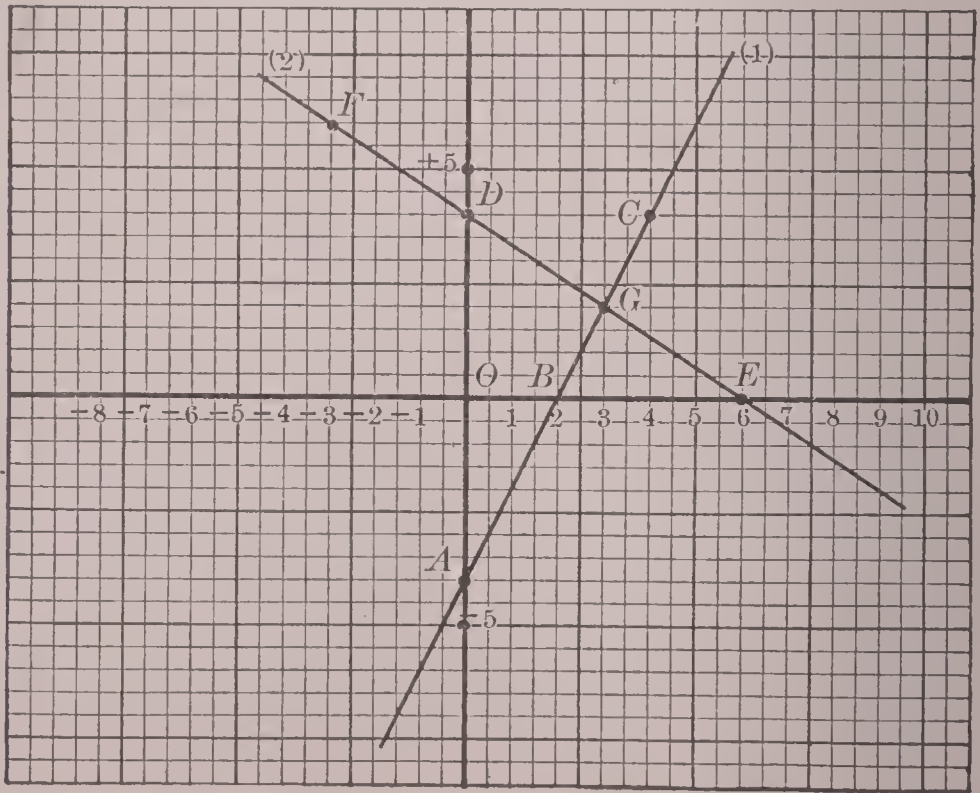
NOTE. — Be careful not to confuse the word *solution* as just used with the word "solution" referring to the process of solving a problem or an example.

Example. — Draw upon the same set of axes the graphs of

$$\begin{cases} 2x - y = 4 \\ 2x + 3y = 12 \end{cases}$$

Solution. — 1. Draw the graphs of the two equations.

(1) $2x - y = 4$		(2) $2x + 3y = 12$	
Point A	When $x = 0$, $y = -4$	Point D	When $x = 0$, $y = 4$
Point B	When $x = 2$, $y = 0$	Point E	When $x = 6$, $y = 0$
Point C	When $x = 4$, $y = 4$	Point F	When $x = -3$, $y = 6$



2. Recall that the coördinates of every point on a graph satisfy the equation of that graph. (See § 144 *b*, page 217.)

Observe point G. Its coördinates are $x = 3$, $y = 2$.

(3, 2) satisfy $2x - y = 4$, since $2 \cdot 3 - 2$ does equal 4.

(3, 2) satisfy $2x + 3y = 12$, since $2 \cdot 3 + 3 \cdot 2$ does equal 12.

$\therefore$ the coördinates of G satisfy both equations.

3. Since no other point lies on both lines, no other pair of numbers can satisfy both equations.

$\therefore$ (3, 2) is the only pair of numbers which will satisfy both equations.

4. These equations are called **independent equations** because each has solutions which are not solutions of the other; they are also **simultaneous equations** (a special kind of independent equations) because they do have one common solution.

NOTE. — On this graph, observe the points marked A, B, C, D, E, and F. Also observe that one line is marked (1), and the other (2), meaning that these lines are the graphs of equations (1) and (2) respectively.

Rule. — To determine graphically the common solution of two linear equations having two variables:

1. Draw the graphs of the two equations upon one set of axes. (Use the form of page 220.)

2. Determine the coördinates of the point which is on both graphs. This is the common solution.

3. Check the common solution by substituting it in both equations.

EXERCISE 107

Determine graphically the common solution of:

1. $\begin{cases} x + y = 6 \\ x - y = 2 \end{cases}$

2. $\begin{cases} 2x + y = 9 \\ 3x - y = 1 \end{cases}$

3. $\begin{cases} x + 2y = 11 \\ 3x - y = 5 \end{cases}$

4. $\begin{cases} 3x + y = 3 \\ 2x - 3y = 13 \end{cases}$

5. $\begin{cases} x + y = -2 \\ 3x + 5y = -2 \end{cases}$

6. $\begin{cases} 6x - 5y = -1 \\ 2x + 3y = 9 \end{cases}$

7. *a.* Draw the graphs of $2x - y = 4$ and $6x - 3y = 12$.

b. What happens? How many *common points* do the graphs have? How many *common solutions*, then, do the equations have?

c. Such equations are called **dependent equations.**

8. In the same way, study equations $x + 2y = 6$ and $2x + 4y = 12$.

9. *a.* Draw the graphs of $3x - y = 5$ and $6x - 2y = 18$.

b. What kind of lines are these graphs? Do these equations have a common solution?

c. Are these equations independent? (See Solution, Step 4, page 221.)

d. Are these equations simultaneous?

c. These equations are **inconsistent equations,** another special kind of independent equations.

10. Study in the same manner the equations $2x - 3y = 6$ and $4x - 6y = 30$.

Determine graphically what kind of equations each of the following pairs is. If they are simultaneous, determine their common solution.

11. $\begin{cases} 7x - 2y = 31 \\ 4x - 3y = 27 \end{cases}$

12. $\begin{cases} 3x + 7y = 4 \\ 3x + 7y = -15 \end{cases}$

13. $\begin{cases} 3x + y = 9 \\ 5x + 4y = 22 \end{cases}$

14. $\begin{cases} 3x + y = 11 \\ 5x - y = 13 \end{cases}$

15. $\begin{cases} 4x + 3y = -1 \\ 8x + 6y = -2 \end{cases}$

16. $\begin{cases} 4x - 3y = 5 \\ 8x - 6y = 17 \end{cases}$

17. $\begin{cases} x - 6y = 2 \\ -8x + 3y = 29 \end{cases}$

18. $\begin{cases} 4x - 2y = 1 \\ 6x + 4y = 19 \end{cases}$

Note. — Sometimes it is difficult to find the common solution accurately by this graphical method. Example 18 may have caused you some difficulty. In the next chapter, other methods of solution are taught.

147. Summary.

a. **Dependent equations** are two equations having the same two (or more) variables, such that every solution of each equation is a solution of the other. They have the same graph.

(See Examples 7 and 8, page 222.) One equation can be changed into the other by multiplying both sides of the equation by the same number.

b. **Independent equations** are two equations having the same two (or more) variables, such that each equation has solutions which are not solutions of the other.

There are two kinds of independent equations.

1. **Simultaneous equations** are two independent equations which have one (or more) common solutions.

If they are linear equations, their graphs are two different straight lines which intersect at one and only one point. (See Example, page 220.) They have only one common solution.

2. **Inconsistent equations** are two independent equations which do not have any common solution.

If they are linear equations, their graphs are two parallel straight lines. (See Examples 9 and 10, page 222.)

EXERCISE 108

Supplementary Examples

1. Draw a bar graph (page 208) representing the strengths of various woods.

WOOD	POPLAR	YELLOW PINE	WHITE PINE	RED FIR	WALNUT	WHITE OAK	MAPLE
NO. OF LB. TO EACH SQ. IN.	7,000	11,000	15,000	10,000	12,000	14,500	10,500

Suggestion. — Let $\frac{1}{4}$ in. = 1000 lb., or let 4 spaces = 1000 lb.

2. Make a table like the one in Example 7, page 213, giving the *simple interest* on \$1 at 4% in 1, 2, 3 . . . 10 years. Draw the graph for the table you have made.

3. Table of distances through which an object, falling freely, falls in various times.

In	0 sec.	1 sec.	2 sec.	3 sec.	4 sec.	5 sec.
It falls	0 ft.	16 ft.	64 ft.	144 ft.	256 ft.	400 ft.

Represent this relation graphically.

Hint. — Let 1 space vertically represent 8 feet; 5 spaces horizontally represent 1 second.

4. Table of relation between Centigrade and Fahrenheit readings.

Centigrade	− 15°	− 5°	0°	10°	20°	30°	40°
Fahrenheit	5°	23°	32°	50°	68°	86°	104°

a. Represent this relation graphically.

Hint. — Horizontally, let 1 space represent 1° Centigrade. Vertically, let 1 space represent 2° or 4° Fahrenheit.

b. From your graph, find the Fahrenheit temperature corresponding to 5° Centigrade; to − 10° Centigrade; to 80° Fahrenheit.

5. Draw the graph of $3V - 2W = 6$.

Hint. — Let the horizontal axis be the V-axis; and the vertical axis be the W-axis.

6. $S = 1.5C + 2$ is the formula for the selling price of an article costing C dollars, so that the gain is \$2 more than 50% of the cost.

a. Draw the graph of this equation, letting the horizontal axis be the C-axis.

b. From your graph, find S when $C = 5.5$.

XII. SIMULTANEOUS LINEAR EQUATIONS *

148. A **linear equation or first degree equation** having two or more unknowns is one in which no unknown appears in the denominator of a fraction, and in which the sum of the exponents of the unknowns in each term is only 1.

Thus, $x + y = 10$ is a linear or first degree equation.

But $xy + x = 10$ is not a first degree equation since the sum of the exponents of x and y in the first term is $1 + 1$ or 2.

Also $\frac{1}{x} + y = 10$ is not a first degree equation in x and y because x appears in a denominator. When the equation is cleared of fractions, $1 + xy = 10x$, and this again is of degree two.

149. A **solution** of an equation having two or more unknowns is a set of values of the unknowns which together satisfy the equation.

Thus, in the equation $x + y = 10$, if $x = 3$, then $3 + y = 10$, and therefore $y = 7$. $x = 3$, $y = 7$ is a solution of $x + y = 10$.

Similarly, in the equation $x + y = 10$

If $x =$	1	2	4	7	10	15	-2	etc.
then $y =$	9	8	6	3	0	-5	$+12$	etc.

Each of these pairs of numbers is a solution of $x + y = 10$.

An equation with two, or more, unknowns has an **infinite number of solutions.** For each value of one unknown, there is a corresponding value of the other unknown. *As one unknown varies* (changes in value), *the other also varies.* The unknowns, therefore, are called **variables** (changing numbers).

Therefore **these equations are called,** preferably, **equations having two, or more, variables.**

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* If you studied Chapter XI, you have had the definitions that appear in §§ 148–151.

150. Two linear equations having two variables are **independent** if each has solutions which are not solutions of the other.

Thus, $x + y = 10$ has the solution $x = 2$, $y = 8$.

$2x + y = 14$ does not have the solution $x = 2$, $y = 8$, since $2 \cdot 2 + 8 = 12$.

Therefore, $x + y = 10$ and $2x + y = 14$ are independent.

They are said to form a **system of independent equations.**

Two independent equations often do have common solutions. They are then called simultaneous equations.

Thus, $x + y = 10$ has the solution $x = 4$, $y = 6$.
Also, $2x + y = 14$ has the solution $x = 4$, $y = 6$.

Simultaneous linear equations have only one common solution.

151. **To solve a system of simultaneous equations** is to find their common solution.

One way to find the common solution of two equations would be to write down different solutions of each, hoping to obtain the common solution by trial.

Thus, $x + y = 10$ has the solutions

$x =$	1	2	3	4	5	6	7	8	etc.
$y =$	9	8	7	6	5	4	3	2	etc.

$x - y = 2$ has the solutions

$x =$	1	2	3	4	5	6
$y =$	-1	0	1	2	3	4

$x = 6$, $y = 4$ is the common solution.

Much better ways to find the common solution are given below.

152. Elimination by addition or subtraction.

Example 1. — Solve the system

$$\begin{cases} 5x - 3y = 19 & (1) \\ 7x + 4y = 2 & (2) \end{cases}$$

Solution. — 1. M_4* (1) $20x - 12y = 76.$ (3)
2. M_3 (2) $21x + 12y = 6.$ (4)
3. Add (3) and (4) $41x = 82.$ (5)
4. D_{41} (5) $x = 2.$ (6)
5. Substitute this value of x in (1) $10 - 3y = 19.$ (7)
$-3y = 9$, or $y = -3.$ (8)
6. The common solution is $x = 2, y = -3.$

Check. — Substitute in (1) $10 + 9 = 19.$
Substitute in (2) $14 - 12 = 2.$

Notice. — 1. That the coefficients of x and y were not the same in equations (1) and (2).

2. That equations (1) and (2) were multiplied by 4 and 3 respectively to make the coefficients of y the same except for sign, in equations (3) and (4).

3. That the terms $12y$ were made to disappear in Step 3 by adding equations (3) and (4). This is allowable because y represents in (3) and (4) the common value of y which we are seeking. y is said to be **eliminated.** This is an example of elimination by addition.

4. If, in equations (3) and (4) the signs of $12y$ were the same (for example, both +) then, in Step 3, we should have subtracted, in order to eliminate $12y$.

Rule. — To solve a system of two simultaneous linear equations having two variables by the addition or subtraction method of elimination:

1. Multiply, if necessary, both the first and second equations by such numbers as will make the coefficients of one of the variables of equal absolute value.

2. If the coefficients have the same sign, subtract one equation from the other; if they have opposite signs, add the equations.

3. Solve the equation resulting from Step 2 for the other variable.

4. Substitute the value of the variable found in Step 3 in any equation containing both variables, and solve for the remaining variable.

5. Check the solution by substituting it in both of the original equations.

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* Read this "multiply both members of equation (1) by 4."

Historical Note. — Little progress was made in solving linear equations having more than one unknown until the latter part of the 15th century, although mathematicians before that time had considered such problems. After Stifel and Stevin had introduced somewhat simple notations for several unknowns and their powers, definite methods for solving equations of the first degree with two unknowns were developed. Johannes Buteo, a French monk (1492–1572), solved equations with three unknowns in a text on algebra which appeared in 1559.

EXERCISE 109

Solve the following systems of equations by the addition or subtraction method of elimination:

1. $\begin{cases} 2x + y = 7 \\ 3x - y = 3 \end{cases}$

2. $\begin{cases} 5a + 2b = 13 \\ 3a + b = 7 \end{cases}$

3. $\begin{cases} m - 3n = -12 \\ 3m + 2n = 19 \end{cases}$

4. $\begin{cases} 3r + 4s = 6 \\ r - 2s = -8 \end{cases}$

5. $\begin{cases} 2c - 3d = 18 \\ 3c + 2d = 14 \end{cases}$

6. $\begin{cases} 7t + 5w = 3 \\ 2t + w = 0 \end{cases}$

7. $\begin{cases} 10x - 3y = -7 \\ 4x + 5y = 22 \end{cases}$

8. $\begin{cases} 9r + 7s = 17 \\ 6r - 5s = -8 \end{cases}$

9. $\begin{cases} 8t + 18w = -15 \\ 3t - 4w = -11 \end{cases}$

10. $\begin{cases} 16m + 3n = 11 \\ 20m = 30 - 7n \end{cases}$

11. $\begin{cases} 11x = 8y + 13 \\ 5x = 7y + 16 \end{cases}$

12. $\begin{cases} 2r = 3s \\ 10r + 6s = 7 \end{cases}$

13. $\begin{cases} 4t + 12w = 7 \\ 6t - 8w = -9 \end{cases}$

14. $\begin{cases} 5m + 11n = 47 \\ 6m + 5n = -1 \end{cases}$

15. $\begin{cases} 15c = 8 - 4d \\ 20c - 12d = 15 \end{cases}$

16. $\begin{cases} 8a + 5b = 26 \\ 3a = 31 - 4b \end{cases}$

17. $\begin{cases} 18p + 25q = -20 \\ 12p + 15q = -14 \end{cases}$

18. $\begin{cases} \frac{1}{2}z + \frac{2}{3}w = 8 \\ \frac{2}{3}z - w = -3 \end{cases}$

19. $\begin{cases} \frac{2}{3}A - \frac{1}{2}B = 8 \\ \frac{5}{6}A + \frac{3}{4}B = -1 \end{cases}$

20. $\begin{cases} 5x + 6y = 1.97 \\ 4x - 5y = .4 \end{cases}$

21. $\begin{cases} \frac{4}{3}s + 2t = -5 \\ \frac{5}{6}s + \frac{4}{9}t = -4\frac{1}{3} \end{cases}$

22. $\begin{cases} 6a + 10b = -1.9 \\ 9a + 100b = 15 \end{cases}$

153. Elimination by substitution is a second common method of solving simultaneous equations.

Example. — Solve the system of equations

$$\begin{cases} 7x - 9y = 15 & (1) \\ 8y - 5x = -17 & (2) \end{cases}$$

Solution. — 1. Solve equation (1) for x in terms of y:

$$7x = 15 + 9y; \quad \therefore\ x = \frac{15 + 9y}{7}. \qquad (3)$$

2. Substitute this value of x in equation (2).

Then $$8y - \frac{5(15 + 9y)}{7} = -17. \qquad (4)$$

3. M_7 (4) $$56y - 5(15 + 9y) = -119.$$

4. $$\therefore\ 56y - 75 - 45y = -119.$$

5. C. T.; A_{75} $$11y = -44.$$

6. D_{11} $$y = -4.$$

7. Substituting this value of y in equation (1),

$$7x + 36 = 15;\ 7x = -21;\ x = -3.$$

8. $\therefore$ the common solution is $x = -3$, $y = -4$.

Check. — In (1), does $-21 + 36 = 15$? Yes.

In (2), does $-32 + 15 = -17$? Yes.

NOTE. — In Step 3, when multiplying $\frac{5(15 + 9y)}{7}$ by 7, we obtain $7 \cdot \frac{5(15 + 9y)}{7}$, or $5(15 + 9y)$.

Rule. — To solve a system of two simultaneous linear equations having two variables by the substitution method of elimination:

1. Solve one equation for one variable in terms of the other variable.

2. Substitute for this variable in the other equation the value found for it in Step 1.

3. Solve the equation resulting in Step 2 for the second variable.

(Continued on page 230.)

4. Substitute the value of the second variable, obtained in Step 3, in any equation containing both variables and solve for the first variable.

5. Check the solution by substituting it in the original equations.

Historical Note. — The earliest use of the Substitution Method of Elimination in print, of which we have any record, is in Newton's Arithmetica Universalis, in 1707.

EXERCISE 110

1. $\begin{cases} 3x + 2y = 16 \\ 7x + y = 19 \end{cases}$

2. $\begin{cases} a - 5b = -2 \\ 9a + 7b = 34 \end{cases}$

3. $\begin{cases} 3c + d = 6 \\ 7c + 5d = -2 \end{cases}$

4. $\begin{cases} 3r + 7s = -5 \\ 6r - 5s = 28 \end{cases}$

5. $\begin{cases} 9t + 4w = -7 \\ 11t + 8w = 7 \end{cases}$

6. $\begin{cases} 2x + 11z = 1 \\ z + 7 - x = 0 \end{cases}$

7. $\begin{cases} 10A = 3B - 1 \\ 7A + 2B = 28 \end{cases}$

8. $\begin{cases} 9m - 14n = -8 \\ m = 2n \end{cases}$

9. $\begin{cases} 12p - 5q = -6 \\ 8p + q = -30 \end{cases}$

10. $\begin{cases} 18w - 4r = -7 \\ r - 6w = 1 \end{cases}$

11. $\begin{cases} 13x - 12t = 35 \\ x = 9t \end{cases}$

12. $\begin{cases} 9x + 4y = -21 \\ 7x - 3y = 2 \end{cases}$

13. $\begin{cases} 7r + 8s = 36 \\ 11r + 5s = -4 \end{cases}$

14. $\begin{cases} 3a + 2b = 1 \\ 12a - 10b = 13 \end{cases}$

15. $\begin{cases} 5t + 2s = 1 \\ 12t + 5s = 1 \end{cases}$

16. $\begin{cases} 16M + 9N = 24 \\ 4M + 3N = 7 \end{cases}$

17. $\begin{cases} 12X - Y = 14 \\ 18X + 5Y = -5 \end{cases}$

18. $\begin{cases} 5c + 10d = 7 \\ 7c - 2d = 5 \end{cases}$

19. $\begin{cases} 4p - 5q = 4 \\ 8p - 15q = 2 \end{cases}$

20. $\begin{cases} 10n - 11z = -4 \\ 5n + 5z = 4 \end{cases}$

21. $\begin{cases} 16x + 9w = 8 \\ 8x - 3w = 9 \end{cases}$

22. $\begin{cases} 5z - 3y = 3 \\ 40z + 6y = 3 \end{cases}$

154. Simultaneous fractional equations.

Example. — Solve the system of equations

$$\begin{cases} \dfrac{7}{x+3} - \dfrac{3}{y+4} = 0 & (1) \\ x(y-2) - y(x-5) = -13 & (2) \end{cases}$$

Solution. — 1. Simplify equation (1)

$$\frac{7}{x+3} - \frac{3}{y+4} = 0.$$

$M_{(x+3)(y+4)}$ $\quad 7(y+4) - 3(x+3) = 0.$

$$7y + 28 - 3x - 9 = 0.$$

$$7y - 3x + 19 = 0, \text{ or } \boxed{3x - 7y = 19} \quad (3)$$

2. Simplify equation (2)

$$x(y-2) - y(x-5) = -13.$$

$$xy - 2x - xy + 5y = -13.$$

$$-2x + 5y = -13, \text{ or } \boxed{2x - 5y = 13} \quad (4)$$

3. Therefore equations (1) and (2) become equations (3) and (4).

$$3x - 7y = 19. \quad (3)$$

$$2x - 5y = 13. \quad (4)$$

4. M_5 (3) $\quad 15x - 35y = 95. \quad (5)$

M_7 (4) $\quad 14x - 35y = 91. \quad (6)$

5. (5) − (6) $\quad x = 4.$

6. Substitute in (4) $\quad 8 - 5y = 13;\ -5y = 5;\ y = -1.$

Check by substituting in (1) and (2).

Equations (3) and (4) of this solution are called the standard form of equations (1) and (2).

Rule. — To solve complicated simultaneous linear equations having two variables:

1. Reduce the equations to standard form by clearing of fractions (when necessary) and simplifying.

2. Solve the resulting system of equations by either method of elimination.

3. Check by substitution in the original equations.

EXERCISE 111

Solve the following systems, some by the addition or subtraction method of elimination, and some by the substitution method:

1. $\begin{cases} \dfrac{3x}{2} - \dfrac{4y}{3} = -1 \\ \dfrac{2x}{3} - \dfrac{y}{4} = \dfrac{7}{12} \end{cases}$

2. $\begin{cases} \dfrac{a+3}{4} - \dfrac{b}{2} = -\dfrac{3}{2} \\ \dfrac{a}{2} - \dfrac{b+3}{4} = -\dfrac{3}{2} \end{cases}$

3. $\begin{cases} 3d - \dfrac{c-4}{6} = 13 \\ 4c + \dfrac{2d-3}{5} = -7 \end{cases}$

4. $\begin{cases} \dfrac{1}{m-1} = \dfrac{1}{n+6} \\ \dfrac{m-5}{n+1} = \dfrac{2}{3} \end{cases}$

5. $\begin{cases} \dfrac{r}{6} - \dfrac{s}{4} = \dfrac{5}{12} \\ \dfrac{3+2r}{5} = \dfrac{7+3s}{10} \end{cases}$

6. $\begin{cases} 5x - \dfrac{y}{2} = -\dfrac{1}{2} \\ \dfrac{4x+1}{6} = \dfrac{y-4}{4} \end{cases}$

7. $\begin{cases} \dfrac{3z-1}{4} - \dfrac{2w+1}{2} = 1 \\ \dfrac{3z+1}{2} - \dfrac{2w-1}{3} = 1 \end{cases}$

8. $\begin{cases} \dfrac{2t+s}{t-s} = \dfrac{1}{2} \\ \dfrac{4-t-s}{2t} = \dfrac{2}{5} \end{cases}$

9. $\begin{cases} \dfrac{A+B-2}{A-B} = \dfrac{1}{5} \\ \dfrac{A+2B-3}{A-3B} = -\dfrac{1}{8} \end{cases}$

10. $\begin{cases} \dfrac{3x-4y}{2} - \dfrac{x+8y}{5} = 3 \\ \dfrac{2x+y}{5} - \dfrac{4x-3y}{9} = 0 \end{cases}$

11. $\begin{cases} \dfrac{w-2}{w+1} = \dfrac{z+2}{z+1} \\ \dfrac{w-3}{w-4} = \dfrac{z+5}{z+4} \end{cases}$

12. $\begin{cases} \dfrac{a-3b-1}{2a-3b-1} = 1 \\ \dfrac{5-a+6b}{7a-9b-1} = 1 \end{cases}$

13. $\begin{cases} \dfrac{X-Y}{4} - \dfrac{X+Y}{5} = 3 \\ \dfrac{2X-Y}{6} - \dfrac{2X+Y}{3} = 1 \end{cases}$

14. $\begin{cases} \dfrac{3t-1}{2} - \dfrac{2r-1}{1} = 0 \\ \dfrac{3t+1}{2} - \dfrac{2(r+1)}{3} = 0 \end{cases}$

15. $\begin{cases} \dfrac{w}{15} - \dfrac{c}{10} = \dfrac{1}{6} \\ \dfrac{3 + 2w}{7 - 3c} = \dfrac{1}{2} \end{cases}$

16. $\begin{cases} \dfrac{1}{4y - x} = \dfrac{2}{4y + x} \\ \dfrac{1}{8y - 5} = \dfrac{3}{8x - 5} \end{cases}$

17. $\begin{cases} \dfrac{M - N - 5}{4} = \dfrac{3M - N}{16} \\ \dfrac{M + 5}{3} - \dfrac{M + N}{8} = \dfrac{11}{6} \end{cases}$

18. $\begin{cases} \dfrac{3a + 5b}{2} - \dfrac{4a}{3} = \dfrac{17}{6} \\ \dfrac{a - 3b}{14} + \dfrac{2a + b}{7} = \dfrac{1}{10} \end{cases}$

19. $\begin{cases} \dfrac{2p + q}{2} - \dfrac{4p - 3q}{5} = \dfrac{7}{15} \\ \dfrac{p + 3q}{5} - \dfrac{3p - q}{7} = \dfrac{2}{15} \end{cases}$

20. $\begin{cases} \dfrac{5x + 4y}{2} - \dfrac{3x - 2y}{3} = \dfrac{11}{10} \\ \dfrac{x + 4y}{2} + \dfrac{x - 4y}{4} = \dfrac{9}{20} \end{cases}$

155. Dependent equations.

Review the definition given in § 150.

Dependent equations are two equations which have exactly the same solutions. For all practical purposes they are the same equation. One can be obtained from the other by multiplying it or by dividing it by the same number.

Thus, $x + y = 10$ and $2x + 2y = 20$ are identical equations. $x = 3$, $y = 7$ is a solution of each. Any other solution of the one is also a solution of the other.

If a system consists of two dependent equations, the system cannot be solved, because actually there is only one equation.

Thus, consider $\begin{cases} x + y = 10 & (1) \\ 2x + 2y = 20 & (2) \end{cases}$

If we try to eliminate by substituting $(10 - x)$ for y in (2), we get

$$2x + 20 - 2x = 20, \text{ or } 0 = 0.$$

There is no opportunity to determine x.

If you studied § 147, review it at this time.

156. Inconsistent equations are two independent equations which do not have any common solution.

Thus, $x + 2y = 7$ and $4x + 8y = 40$ are inconsistent.
If $4x + 8y = 40$ is divided by 4, we get $x + 2y = 10$.
Clearly, if $x + 2y = 7$, $x + 2y$ cannot $= 10$.

Notice that the coefficients 4 and 8 of the second equation are 4 times the corresponding coefficients of the first equation, but that 40 is not 4 times 7. This is characteristic of inconsistent equations.

If we try to solve the system

$$\begin{cases} x + 2y = 7 \\ 4x + 8y = 40 \end{cases}$$

by the subtraction method, we get

$$\begin{cases} 4x + 8y = 28 \\ \underline{4x + 8y = 40} \end{cases}$$

Subtracting, $0 = -12$

This is impossible.

157. When solving problems by means of two or more literal numbers, as many independent equations must be obtained from the conditions of the problem as there are letters used.

NOTE. — Most problems that can be solved by means of equations having two unknowns, can also be solved by an equation having only one unknown. Nevertheless, use two unknowns when solving the following problems.

EXERCISE 112

1. Find two numbers whose sum is 20 and whose difference is 7.

Solution. — 1. Let l = the larger number and s = the smaller number.

(Complete the solution.)

2. Separate 39 into two parts such that 4 times the smaller exceeds the larger by 6.

Solution. — 1. Let l = the larger part and s = the smaller part of 39.

2. $l + s = ?$

3. Also $4s - l = ?$

(Complete the solution.)

3. *a.* Solve Example 2 by an equation having only one unknown.

b. Which solution do you prefer?

4. If 2 be added to both numerator and denominator of a certain fraction, the resulting fraction equals $\frac{1}{3}$; if 7 be added to both terms of the original fraction, the result equals $\frac{1}{2}$. What is the fraction?

Solution. — 1. Let n = the numerator and

d = the denominator of the fraction.

$$\therefore \frac{n}{d} = \text{the fraction.}$$

2. Then,
$$\frac{n+2}{d+2} = \frac{1}{3}. \qquad (1)$$

3. Also
$$\frac{n+7}{d+7} = \frac{1}{2}. \qquad (2)$$

Reduce equations (1) and (2) to the standard from; complete and check the solution.

5. Find two numbers whose sum is 56, and such that $\frac{1}{2}$ of the greater exceeds $\frac{3}{4}$ of the smaller by 3.

6. Solve Problem 5, using only one unknown.

7. The larger of certain two numbers exceeds twice the smaller by 8. Five times the smaller exceeds twice the larger by 5. What are the numbers?

8. Separate 31 into two parts such that the quotient of the larger divided by the smaller is 3, and the remainder is 7.

9. If 2 be added to the numerator of a certain fraction, the resulting fraction equals $\frac{1}{2}$. If 1 be added to the denominator of the original fraction, the resulting fraction equals $\frac{1}{3}$. What is the fraction?

10. A's age now exceeds twice B's age by 3. Three years ago, A was 4 times as old as B was then. What are their ages now?

11. The perimeter of a certain rectangle is 72 inches. The width exceeds $\frac{2}{3}$ of the length by 1 inch. What are the length and width?

12. In an isosceles triangle, two angles always are equal, and the sum of all the angles is 180°. In a certain isosceles triangle, each of the equal angles is 15° more than one half the third angle How large is each angle of the triangle?

13. Twice the shorter side of a certain parallelogram exceeds the longer side by 6 inches. One half the longer side exceeds one seventh the shorter side by 9 inches. How long are the sides?

14. If three times the smaller of certain two numbers be divided by the larger, the quotient is 2 and the remainder is 3. The larger exceeds the smaller by $\frac{7}{6}$. What are the numbers?

15. A purse contained $5.45 in quarters and dimes. Payment for a 5¢ purchase was made with a quarter, and dimes were received in change. Then the number of dimes was twice the number of quarters remaining. How many dimes and quarters were there at first?

16. Six years ago, A was 12 times as old as B was; three years from now, he will be 3 times as old as B is then. What are their ages?

17. Two angles are complementary. (See page 23.) One sixth of the larger exceeds one eighth of the smaller by $4\frac{1}{2}°$. How large is each angle?

18. The supplement of a certain angle is $\frac{2}{3}$ of the angle. How large is each?

19. If the numerator of a certain fraction be doubled and the denominator be increased by 5, the resulting fraction equals $\frac{7}{8}$. If the numerator be decreased by 1, and the denominator be increased by 7, the resulting fraction has the value $\frac{1}{3}$. What is the given fraction?

20. The length of a certain room is 7 feet more than its width. If the room is made 1 foot wider and 3 feet longer, the area is increased by 66 square feet. What were the original dimensions?

Solution. — 1. Let $l =$ the no. of feet in the length;

2. and $w =$ the no. of feet in the width.

3. $\therefore \boxed{l = w + 7}$ (1)

4.

	LENGTH	WIDTH	AREA
Old dimensions	l	w	lw
New dimensions	$l + 3$	$w + 1$	$(l + 3)(w + 1)$

5. $\therefore \boxed{(l + 3)(w + 1) = lw + 66}$ (2)

(Complete the solution, using equations (1) and (2).)

21. If 5 feet be added to the width and 15 feet to the length of a certain lot, the area is increased 1750 sq. ft. If 10 feet be subtracted from the length and 15 feet be added to the width, the area is increased 750 sq. ft. Find the length and width.

22. One sum of money is invested at 6%. A second sum, which exceeds the first sum by $150, is invested at 7%. The interest on the second sum exceeds the interest on the first sum by $19. What are the two sums?

23. Solve Example 22 by means of only one unknown.

24. The total value of two investments is $4500. The annual interest earned on one is 5%, and on the other is 6%. The total interest is $252. What are the two sums?

25. The simple interest on $1200 at 7% for a certain number of years exceeds the simple interest on $750 at 6% for a second period of years by $27. The second period of years exceeds the first by 2. How long is each period of years?

26. A crew can row $10\frac{1}{2}$ miles downstream in one hour and $5\frac{1}{2}$ miles upstream in the same time. What is the rate of the current of the river, and of the crew in still water?

27. It took a motor boat $1\frac{1}{3}$ hr. to go 20 miles downstream, and $2\frac{2}{9}$ hr. to return. What was its own rate in still water and what was the rate of the current of the river?

28. Some boys were rowing on a river. It took them only three hours to go 20 miles downstream; but it took 10 hours to return. What was their own rate of rowing in still water and what was the rate of the current?

29. On a river whose current is known to be $2\frac{1}{4}$ miles per hour, some boys rowed downstream for 2 hours. It took them 6 hours to return. How far did they go, and what was their rate of rowing in still water?

30. If a certain lot is made 5 feet longer and 2 feet wider, its area is increased 415 square feet. If it is made 3 feet narrower and 10 feet longer, its area is decreased 60 square feet. What are its dimensions?

158. Relations among digits of a number. Integers are written by means of the **digits** 0, 1, 2, 3, . . . 9.

Thus, 38 is a number of 2 digits; 3 represents 3×10 units; 8 represents 8 units. The sum of the digits is $3 + 8$ or 11.

If t is the tens' digit, and u is the units' digit, the number contains $10t + u$ units. The sum of the digits is $t + u$.

When the digits are reversed, a new number is formed. Thus, reversing the digits of 52 gives 25.

$$\text{Notice that } 52 = (5 \times 10 + 2) \text{ units,}$$
$$\text{and that } 25 = (2 \times 10 + 5) \text{ units.}$$

Similarly, if x and y are the tens' and units' digits of a number, then the number contains $10x + y$ units. If the digits are reversed, the new tens' digit is y and the units' digit is x; then the new number contains $(10y + x)$ units.

EXERCISE 113

1. How many units are there in the number:

a. whose tens' digit is 8 and units' digit is 5?

b. whose tens' digit is a and units' digit is b?

c. whose tens' digit is b and units' digit is a?

2. The hundreds', tens', and units' digits of a certain number are x, y, and z respectively.

a. How many units does this number contain?

b. If the digits are reversed, how many units does the new number contain?

3. Let t represent the tens' digit and u the units' digit of a certain number of two digits.

a. Represent the difference of the digits. (See § 95, page 123.)

b. Represent the number itself.

c. Represent the quotient of the tens' divided by the units' digit.

d. Represent the number with digits in the reverse order.

e. Represent this last number divided by the sum of the digits of the given number.

4. The sum of the digits of a certain two-digit number is 11. If the digits be reversed, and the new number be divided by the old, the quotient is 2 and the remainder is 7. What is the number?

Solution. — 1. Let t = the tens' digit of the number,
and u = the units' digit of the number.

$$\therefore \boxed{t + u = 11} \qquad (1)$$

2. Then

	TENS' DIGIT	UNITS' DIGIT	NUMBER ITSELF
Given number	t	u	$10\,t + u$
New number	u	t	$10\,u + t$

$$\therefore (10\,u + t) = 2(10\,t + u) + 7. \qquad (2)$$

(Complete the solution, using equations (1) and (2). Check the solution, by reference to the statements in the given problem.)

5. The tens' digit of a certain two-digit number exceeds its units' digit by 5. If the digits be reversed, the new number is one fourth of the result obtained by adding 3 to the given number. What is the number?

6. The tens' digit of a certain two-digit number exceeds twice the units' digit by 1. If the digits be reversed, the sum of the new number and the original number is 143. What is the number?

7. The sum of the two digits of a certain two-digit number is 9. If the digits be reversed, the resulting number is 9 less than 3 times the original number. What is the number?

8. A certain number consists of two digits. If the digits be reversed, the sum of twice the original number and the new number is 168. Also the original number exceeds 4 times the sum of its digits by 3. What is the number?

9. A certain number consists of three digits of which the units' digit is 5. The sum of the digits is 12. If the hundreds' and tens' digits are reversed, the new number is 90 less than the original number. What is the number?

10. In a certain number consisting of three digits, the tens' digit is 6. The hundreds' digit is 1 less than twice the units' digit. If the hundreds' and units' digits be interchanged, the new number exceeds one half the original number by 85. What is the number?

EXERCISE 114

Review Examples

1. Divide $4m^4 - 9m^2 + 6m - 1$ by $2m^2 + 3m - 1$. Check, letting $m = 2$.

2. Reduce to lowest terms $\frac{3c^2 - 6c}{6c^2 - 24}$.

3. Solve for k: $\frac{7k}{5} - \frac{1}{14}(k - 11) = \frac{3}{7}(k - 25) + 34$.

4. *a.* Solve $T = 2\pi R(R + H)$ for H in terms of T, π, and R.

b. In your result, let $\pi = 3.14$, $R = 10$, $T = 794.42$. Find H to two decimal places.

5. *a.* The three digits of a number are a, b, and c. Represent the number formed by reversing the digits.

b. A man earns e dollars a month, and spends s dollars a month. How much will he have saved in 3 years?

c. John is y years old now. Write the equation expressing the fact that 4 times his age 3 years ago is twice his age 5 years from now.

6. *a.* Simplify $\frac{a+1}{a^2-a-6} - \frac{a-4}{a^2-4a+3} + \frac{a+3}{a^2+a-2}$.

b. Check the solution by letting $a = -1$.

7. Simplify $\frac{4r^2-12rs+9s^2}{r^2+4rs+4s^2} \cdot \frac{4r^2-4s^2}{4r^2-9s^2} \div \frac{3rs-3s^2}{5r^2+10rs}$.

8. Solve and check $\frac{6x+5}{2x^2-2x} + \frac{2}{x^2-1} = \frac{3x}{x^2-1}$.

9. Solve and check $\frac{x-b}{x-2a} - \frac{x+b}{+2a} = \frac{4a^2}{x^2-4a^2}$.

SUPPLEMENTARY TOPICS

159. Literal simultaneous equations.

Example. — Solve for x and y the system:

$$\begin{cases} ax + by = c & (1) \\ rx + y = t & (2) \end{cases}$$

Solution. — *a.* Eliminate y.

1. M_s (1) $\quad asx + bsy = cs.$ (3)
2. M_b (2) $\quad brx + bsy = bt.$ (4)
3. (3) − (4) $\quad asx - brx = cs - bt.$ (5)
4. Factoring, $\quad x(as - br) = cs - bt.$
5. $\text{D}_{(as-br)}$ $\quad x = \frac{cs - bt}{as - br}.$

b. Next, going back to (1) and (2), eliminate x.

6. M_r (1) $\quad arx + bry = cr.$ (6)
7. M_a (2) $\quad arx + asy = at.$ (7)
8. (6) − (7) $\quad bry - asy = cr - at.$ (8)
9. Factoring, $\quad y(br - as) = cr - at.$
10. $\text{D}_{(br-as)}$ $\quad y = \frac{cr - at}{br - as}$, or $y = \frac{at - cr}{as - br}.$
11. $\therefore$ The common solution is $x = \frac{cs - bt}{as - br}$; $y = \frac{at - cr}{as - br}.$

NOTE 1. — Step 3 can be omitted if you notice that asx and brx are *like terms*, and that their difference can be obtained by multiplying their common factor x by the difference of its coefficients; namely, $(as - br)$.

NOTE 2. — Remember that this result is true for all values of a, b, c, r, s, and t which do not make any denominators equal to zero.

EXERCISE 115

Solve the following systems for x and y:

1. $\begin{cases} 2x - 3y = a \\ 3x + 4y = b \end{cases}$

2. $\begin{cases} cx + cy = 2a \\ cx - cy = 2b \end{cases}$

3. $\begin{cases} 4x + ay = b \\ 5x + by = a \end{cases}$

4. $\begin{cases} bx + y = b \\ ax + 2y = a \end{cases}$

5. $\begin{cases} ax + by = 1 \\ x + y = 1 \end{cases}$

6. $\begin{cases} ax - by = 1 \\ mx + ny = 1 \end{cases}$

7. $\begin{cases} ax - by = c \\ dx - ey = f \end{cases}$

8. $\begin{cases} 2x + y = 3a + b \\ x + 2y = 3a + 2b \end{cases}$

9. $\begin{cases} 2x + 3y = 5a + b \\ 3x + 2y = 5a - b \end{cases}$

10. $\begin{cases} mx + y = m^2 + n \\ x + my = m + mn \end{cases}$

11. $\begin{cases} rx + ry = 2r^2 \\ sx - sy = 2s^2 \end{cases}$

12. $\begin{cases} bx + ay = a + b \\ abx + aby = a^2 + b^2 \end{cases}$

13. $\begin{cases} ax = by \\ abx + aby = (a + b)^2 \end{cases}$

14. $\begin{cases} bx - ay = ab \\ 3ax + 3by = 2a^2 - b^2 \end{cases}$

15. *a.* Find two numbers whose sum is m and difference is n.

b. Using the results as formulæ, find the numbers whose sum is 33 and difference is 7.

16. *a.* Separate n into two parts such that the larger will exceed twice the smaller by m.

b. Using these results as formulæ, find the two parts when n is 50 and m is 8.

17. *a.* Separate a into two parts such that when the larger is divided by the smaller the quotient is b and the remainder is c.

b. What are these two parts when $a = 75$, $b = 2$, and $c = 3$?

c. What are these two parts when $a = 100$, $b = 4$, $c = 5$?

18. *a.* A is now r times as old as B. s years ago A was t times as old as B. What are their present ages?

b. What are these ages when $r = 2$, $s = 15$, $t = 5$?

c. These formulæ will always give some sort of results. The results may be impossible because the conditions are impossible. Thus, let $r = 5$, $s = 10$, $t = 3$.

19. *a.* A sum of money amounting to m dollars consists of quarters and dimes alone. The number of quarters is n more than the number of dimes. How many of each coin are there?

b. What are the results when $m = 5$ and $n = 6$?

c. What are the results when $m = 5.6$ and $n = 7$?

d. What are the results when $m = 5$ and $n = 3$?

20. *a.* If m be added to the numerator of a certain fraction, the resulting fraction has the value c. If n be subtracted from the numerator of the fraction, the resulting fraction has the value d. What is the fraction?

b. What is this fraction when $m = 4$, $c = 2$, $n = 2$, and $d = 1$?

21. *a.* The length of a certain rectangle exceeds its width by c feet. The perimeter is p feet. Find its length and width.

b. Using the results of part *a* as formulæ, find the length and width when c is 7 feet and p is 54 feet.

c. Repeat part *b* when c is 8 feet and p is 90 feet.

22. *a.* There are two angles which are complementary. One exceeds the other by n degrees. How large is each?

b. Using the results of part *a* as formulæ, find the two angles when n is 15°.

c. Repeat part *b* when n is 23°.

160. Fractional equations linear in $\frac{1}{x}$ and $\frac{1}{y}$.

Example. — Solve the system
$$\begin{cases} \dfrac{10}{x} - \dfrac{9}{y} = 8. & (1) \\[2ex] \dfrac{8}{x} + \dfrac{15}{y} = -1. & (2) \end{cases}$$

Solution. — 1. M_5 (1) $\quad \dfrac{50}{x} - \dfrac{45}{y} = 40. \qquad (3)$

2. M_3 (2) $\quad \dfrac{24}{x} + \dfrac{45}{y} = -3. \qquad (4)$

3. Adding, $\quad \dfrac{74}{x} = 37. \qquad (5)$

4. $\quad \therefore\ 74 = 37x$, or $x = \mathbf{2}$.

5. Substitute 2 for x in (1) $\quad 5 - \dfrac{9}{y} = 8$.

6. Solving for y, $\quad y = -3$.

EXERCISE 116

Solve the following systems of equations:

1. $\begin{cases} \dfrac{4}{x} + \dfrac{9}{y} = 5 \\[2ex] \dfrac{8}{x} - \dfrac{3}{y} = 3 \end{cases}$

2. $\begin{cases} \dfrac{12}{x} - \dfrac{6}{y} = -1 \\[2ex] \dfrac{9}{x} + \dfrac{10}{y} = -8 \end{cases}$

3. $\begin{cases} \dfrac{4}{x} + \dfrac{3}{y} = 7 \\[2ex] \dfrac{10}{x} - \dfrac{5}{y} = -\dfrac{15}{2} \end{cases}$

4. $\begin{cases} 2y - \dfrac{1}{x} = \dfrac{7}{6} \\[2ex] 3y + \dfrac{4}{x} = -1 \end{cases}$

5. $\begin{cases} \dfrac{5}{x} - \dfrac{8}{y} = 1 \\[2ex] \dfrac{3}{x} + \dfrac{2}{y} = -13 \end{cases}$

6. $\begin{cases} \dfrac{2}{A} - \dfrac{15}{B} = -\dfrac{7}{2} \\[2ex] \dfrac{7}{A} + \dfrac{5}{B} = -\dfrac{3}{4} \end{cases}$

7. $\begin{cases} \dfrac{1}{2m} - \dfrac{3}{n} = -\dfrac{13}{4} \\[2ex] \dfrac{1}{m} + \dfrac{2}{n} = \dfrac{25}{6} \end{cases}$

Hint. — M (1) by 4.

8. $\begin{cases} \dfrac{3}{a} + \dfrac{2}{b} = -10 \\[2ex] \dfrac{1}{3a} - \dfrac{1}{2b} = -\dfrac{35}{12} \end{cases}$

Hint. — M (2) by 12.

9. $\begin{cases} \dfrac{3}{2r} + \dfrac{1}{3s} = -\dfrac{1}{2} \\ \dfrac{5}{r} + \dfrac{3}{2s} = -\dfrac{17}{6} \end{cases}$ **10.** $\begin{cases} \dfrac{4}{3c} + \dfrac{5}{4d} = -\dfrac{11}{2} \\ \dfrac{1}{6c} - \dfrac{2}{5d} = -\dfrac{9}{5} \end{cases}$

Work problems and problems about reciprocals

NOTE. — For work problems, review Examples 1–4, page 192.

The **reciprocal** of 3 is 1 ÷ 3 or $\frac{1}{3}$; of $\frac{3}{2}$ is 1 ÷ $\frac{3}{2}$ or $\frac{2}{3}$; of x is $\frac{1}{x}$.

11. The sum of the reciprocals of certain two numbers is $\frac{1}{2}$. Twice the reciprocal of the smaller, diminished by the reciprocal of the larger, also is $\frac{1}{2}$. What are the numbers?

Suggestion. — If l = the larger, then 1 is its reciprocal.

12. There are two unequal numbers such that the reciprocal of the smaller, diminished by the reciprocal of the larger, is $\frac{2}{3}$. Twice the reciprocal of the smaller plus five times the reciprocal of the larger equals 6. What are the numbers?

13. A and B together do a certain piece of work in 10 days. They can do it also if A works 12 days and B works 6 days. How long would it take each to do it alone?

Suggestion. — If A can do it in a days, how much can he do in 1 day? in 10 days? etc.

14. A and B together can do $\frac{5}{8}$ of a certain piece of work in 3 days. They can do all of it if A works 9 days alone and B works 2 days alone. How long would it take each of them to do all of it alone?

15. A workman and one helper did a certain piece of work in 6 days. On another occasion, the workman and two helpers did the same work in 4 days. If the two helpers worked at the same rate, how long would it take the workman alone or each of the helpers alone to do the work?

161. Equations containing three variables. If there are three variables, there must be three independent equations in the system.

EXERCISE 117

Solve the following systems of equations.

$$\textbf{1.}\quad \begin{cases} 2x - y + z = 5 & (1) \\ 3x + 2y + 3z = 7 & (2) \\ 4x - 3y - 5z = -3 & (3) \end{cases}$$

Solution. — *a.* Eliminate z from (1) and (2).

1. M_3 (1) $\quad 6x - 3y + 3z = 15.$ (4)

2. Subtract (2) from (4) $\quad \boxed{3x - 5y = 8}$ (5)

b. Eliminate z from (1) and (3).

1. M_5 (1) $\quad 10x - 5y + 5z = 25.$ (6)

2. Add (3) and (6) $\quad \boxed{14x - 8y = 22}$ (7)

c. Complete the solution by solving the system consisting of equations (5) and (7), thus finding x and y.

d. Substitute in (1) the values found for x and y. Solve for z.

e. Check by substituting in the three original equations.

$$\textbf{2.}\quad \begin{cases} 2x - y + z = 3 \\ 3x + 2y - z = 4 \\ 4x + 3y - 2z = 4 \end{cases}$$

$$\textbf{3.}\quad \begin{cases} 3x + 4y + z = -1 \\ 2x - y + 2z = -1 \\ x + 3y - z = 2 \end{cases}$$

$$\textbf{4.}\quad \begin{cases} 4x - y + 2z = 1 \\ 3x + 2y + 3z = 4 \\ 5x - 7y = -3 \end{cases}$$

$$\textbf{5.}\quad \begin{cases} 3w + 2r = 9 \\ 2z - r = 11 \\ 5w - 6z = 1 \end{cases}$$

$$\textbf{6.}\quad \begin{cases} 6s + 3t + 2v = 1 \\ 3s - 4v = 4 \\ 5s - t = 14 \end{cases}$$

$$\textbf{7.}\quad \begin{cases} 2A + 3B - C = 0 \\ 4A - 6B + C = 2 \\ A - B + C = 2\frac{1}{6} \end{cases}$$

$$\textbf{8.}\quad \begin{cases} 8m - 3n - 12c = -2 \\ 3m - n - 6c = -\frac{1}{3} \\ 5m + 9n = \frac{1}{2} \end{cases}$$

$$\textbf{9.}\quad \begin{cases} 2x - 2y + 3z = \frac{13}{30} \\ 3x - 2z = \frac{8}{5} \\ 4y + 10z = 5 \end{cases}$$

10. In a certain triangle, twice angle A exceeds angle B by 30°, and twice angle A exceeds angle C by 40°. Find the angles of the triangle. (See page 25, Example 22.)

11. The perimeter of a certain triangle is 165 inches. The sum of the first and second sides equals twice the third side;

twice the first side exceeds the second side by 10 inches. How long are the sides of the triangle?

12. A certain publishing company offered three magazines called *A*, *B*, and *C* at the following combination rates:

For *A* and *B* together	$3.50
For *A* and *C* together	$5.50
For *B* and *C* together	$5.00

What is the price at which each magazine is figured in these combinations?

13. The sum of the three digits of a certain number is 14. The units' digit exceeds the sum of its hundreds' and tens' digit by 2. The tens' digit is one half the units' digit. What is the number?

14. The sum of the three digits of a certain number is 16. If the digits be reversed (see page 238), the new number exceeds the old number by 99. The sum of the tens' digit and units' digit equals 3 times the hundreds' digit. What is the number?

15. The sum of the three digits of a certain number is 7. *If the tens' and hundreds' digits be interchanged,* and the new number be divided by the old, the quotient is 2 and the remainder is 20. The tens' digit exceeds twice the hundreds' digit by 1. What is the number?

16. A and C together can do a piece of work in 4 days. If A and B work together for 3 days, then B can complete the work in 1 more day. If B and C work together for 4 days, C can complete the work in 2 days more. How long would it take each to do it alone?

XIII. SQUARE ROOT AND QUADRATIC SURDS

162. The square root of a perfect square monomial was found by inspection in § 91, page 119. Review this section.

163. Every number has two square roots. They have equal absolute values but opposite signs.

Thus, $\sqrt{a9\ {}^4b^2} = +3\,a^2b$, because $(+3\,a^2b)^2 = 9\,a^4b^2$.
Also, $\sqrt{9\,a^4b^2} = -3\,a^2b$, because $(-3\,a^2b)^2 = 9\,a^4b^2$.

These two roots are written together by means of the **double sign** ±, read *plus or minus*.

Thus, $\sqrt{9\,a^4b^2} = \pm 3\,a^2b$; read *plus or minus* $3\,a^2b$.

Of these two square roots, **the positive one is called the principal square root.**

When the square root is mentioned, usually the principal square root is meant.

164. The square root of a large number often may be obtained by inspection, after factoring the number.

Thus, $\sqrt{1764\,a^4} = \sqrt{4 \cdot 441\,a^4} = \sqrt{4 \cdot 9 \cdot 49\,a^4} = 2 \cdot 3 \cdot 7 \cdot a^2 = 42\,a^2$.

This solution uses the following fundamental principle:

Rule. — The square root of the product of two or more numbers equals the product of their square roots.

Thus, $\sqrt{4 \cdot 25} = 2 \cdot 5 = 10$; $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$.

EXERCISE 118

Find the square roots of:

1. $36\,m^4$
2. $64\,a^6b^2$
3. $121\,m^2n^4$
4. $169\,x^6$
5. $196\,r^2s^4$
6. $225\,x^2$
7. $900\,a^2$
8. $576\,c^4$
9. $1089\,x^4$
10. $1764\,y^6$
11. $1936\,z^2w^2$
12. $2304\,m^6n^4$

13. Find the square roots of $9a^2 - 12ax + 4x^2$.

Solution. — 1. $\sqrt{9a^2 - 12ax + 4x^2} = \sqrt{(3a - 2x)(3a - 2x)}.$

2. $\therefore \sqrt{9a^2 - 12ax + 4x^2} = \sqrt{(3a - 2x)^2}$
$= \pm (3a - 2x).$

14. $25a^2 - 10a + 1$

15. $c^2 - 16c + 64$

16. $m^2 - 20mn + 100n^2$

17. $4a^2 - 20a + 25$

18. $9a^2 - 30ab + 25b^2$

19. $16c^2 - 24cd + 9d^2$

Example **20.** — Make a perfect square trinomial of $25x^2 - 30xy + (?)$ by supplying the missing term; then find the principal square root of the result.

Solution. — 1. In the expressions of Examples 13–19 observe:
Two terms (the first and third) *are perfect squares, preceded by plus signs; the remaining term is twice the product of their square roots.*

2. In this example, $\sqrt{25x^2} = 5x$.

3. $-30xy$ must be $2 \cdot (5x) \cdot$ some number.
$\therefore$ that number must be $3y$, since $30xy \div 10x = 3y$.

4. The remaining term must be $(3y)^2$ or $9y^2$.

5. The perfect square trinomial is $25x^2 - 30xy + 9y^2$

6. and $\sqrt{25x^2 - 30xy + 9y^2} = (5x - 3y)$.

Make perfect square trinomials of the following; then give the principal square root of each:

21. $a^2 + 6a + (?)$

22. $r^2 - 12r + (?)$

23. $t^2 - 18tw + (?)$

24. $m^4 - 24m^2x + (?)$

25. $x^2 - (?) + 36y^2$

26. $x^2 + 3x + (?)$

27. $y^2 - 5y + (?)$

28. $z^2 + \frac{2}{3}z + (?)$

29. $t^2 - \frac{2}{5}t + (?)$

30. $36x^2 - 12x + (?)$

31. $49m^4 - 14m^2n + (?)$

32. $9a^2 - (?) + 36b^2$

33. $49x^2 - 42x + (?)$

34. $36a^2 - 60ab + (?)$

165. **The square root of a perfect square polynomial can be found by a long division process.**

Example 1. — Find the square root of $9x^2 + 25y^2 - 30xy$.

Solution. — 1. Arrange the polynomial in descending powers of x.

2. $\sqrt{9x^2} = 3x$. Write $3x$ in the root.
3. Square $3x$, getting $9x^2$. Write it below $9x^2$.
4. Subtract, obtaining the first remainder.
5. Trial divisor, $2 \cdot 3x = 6x$. $-30xy \div 6x = -5y$. Write $-5y$ in the root and add $-5y$ to $6x$, forming the complete divisor.
6. Multiply the complete divisor by $-5y$.
7. Subtract. In this case, no remainder.

$$\begin{array}{r|l}
\text{Root} & 3x \qquad\qquad -5y \\
\hline
 & 9x^2 - 30xy + 25y^2 \\
 & 9x^2 \\
\hline
6x & -30xy + 25y^2 \\
-5y & \\
\hline
6x - 5y & -30xy + 25y^2 \\
\hline
\end{array}$$

8. The *square roots* are $\pm(3x - 5y)$ or $+(3x - 5y)$, which is $3x - 5y$, and $-(3x - 5y)$, which is $-3x + 5y$.

Note. — All the explanation of this solution should be carried along mentally. Only the solution is written.

Example 2. — Find the principal square root of

$$20x^3 - 70x + 4x^4 + 49 - 3x^2.$$

Solution. — 1. Arrange the polynomial in descending powers of x.

2. $\sqrt{4x^4} = 2x^2$. Write it in root.
3. $(2x^2)^2 = 4x^4$. Write it. Subtract.
4. Trial divisor, $2 \cdot 2x^2$.
5. $20x^3 \div 4x^2 = 5x$. Write it in root.
6. Add $+5x$ to $4x^2$.
7. Multiply $4x^2 + 5x$ by $+5x$. Subtract.
8. Trial divisor, $2(2x^2 + 5x)$.
9. $-28x^2 \div 4x^2 = -7$. Write it in root.
10. Add -7 to $4x^2 + 10x$.
11. Multiply $4x^2 + 10x - 7$ by -7. Subtract.

$$\begin{array}{r|l}
 & 2x^2 + 5x - 7 \\
\hline
 & 4x^4 + 20x^3 - 3x^2 - 70x + 49 \\
 & 4x^4 \\
\hline
4x^2 & +20x^3 - 3x^2 - 70x + 49 \\
+5x & \\
\hline
4x^2 + 5x & 20x^3 + 25x^2 \\
\hline
4x^2 + 10x & -28x^2 - 70x + 49 \\
-7 & \\
\hline
4x^2 + 10x - 7 & -28x^2 - 70x + 49 \\
\hline
\end{array}$$

EXERCISE 119

Find the square root of:

1. $25x^2 + 40xy + 16y^2$
2. $9a^2 - 30ab + 25b^2$
3. $36c^2 - 60cd + 25d^2$
4. $49m^4 - 28m^2n + 4n^2$
5. $25r^2 - 70rs + 49s^2$
6. $4x^4 + 4x^3 + 5x^2 + 2x + 1$
7. $4x^4 - 4x^3 + 5x^2 - 2x + 1$
8. $9x^4 + 6x^3 + 13x^2 + 4x + 4$
9. $9a^4 + 12a^3 - 2a^2 - 4a + 1$
10. $y^4 - 6y^3 + 13y^2 - 12y + 4$
11. $4c^4 - 12c^3d + 17c^2d^2 - 12cd^3 + 4d^4$
12. $9m^4 - 24m^3n + 10m^2n^2 + 8mn^3 + n^4$
13. $1 - 2x - x^2 + 2x^3 + x^4$
14. $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$
15. $4a^2 + 9b^2 + c^2 + 12ab - 4ac - 6bc$
16. $9a^2 - 24ab + 30ac + 16b^2 - 40bc + 25c^2$
17. $4x^2 - 12xy - 16xz + 9y^2 + 24yz + 16z^2$
18. $x^4 + 2x^3 + 2x^2 + x + \frac{1}{4}$
19. $x^4 - x^2y + \frac{2}{3}x^2z + \frac{1}{4}y^2 - \frac{1}{3}yz + \frac{1}{9}z^2$
20. $\frac{1}{4}a^4 - 2a^2b + a^2c + 4b^2 - 4bc + c^2$

166. The square root of an arithmetical number.

Any number having one or two figures to the left of the decimal point (like 1, or 99.5) equals or exceeds 1 and is less than 100; therefore **its square root**, equal to or exceeding 1 and less than 10, **must have one figure to the left of the decimal point.**

Any number having three or four figures to the left of the decimal point (like 100.25 or 9999.7) equals or exceeds 100 and is less than 10,000. Therefore **its square root**, equal to or exceeding 10 and less than 100, **must have two figures to the left of the decimal point.**

Hence, **if the given number is divided into groups of two figures each,** starting at the decimal point, **for each group in the number there will be one digit in the square root.** The groups are called **periods.**

Thus, 2345 becomes 23 45. It has two periods. Its square root has two figures to the left of the decimal point, a tens' figure and a units' figure. 34,038 becomes 3 40 38. It has three periods. Its square root has three figures to the left of the decimal point. The left-most period may contain only one figure, as in this case.

A decimal is divided in the same manner, counting in both directions from the decimal point. Thus, 34256.895 becomes 3 42 56.89 50. The square root of this number has three figures to the left of the decimal point and two to the right.

The square root is found as follows:

Example 1. — Find the square root of 4624.

Solution. — 1. 4624 must be the square of a number more than 60 and less than 70, since $60^2 = 3600$ and $70^2 = 4900$.

That is, the square root is 60 + some number less than 10.

		a		*b*
		$6\cancel{0} + 8.$		6 8.
$60^2 = 3600$. Subtract from 4624.		46 24.		46 24.
2. Trial divisor, $2 \times 60 = 120$.		36 $\cancel{0}\cancel{0}$		36
3. $1024 \div 120 = 8^+$. Add 8 to 120.	120	10 24	120	10 24
Add 8 to 60 in the root.	8		8	
4. Multiply 128 by 8. Subtract.	128	10 24	128	10 24

Note. — The solution is always written in the form *b*. The canceled zeros in solution *a* are omitted in solution *b*.

Rule. — To find the square root of a number:

1. Separate the number into groups of two figures each, starting at the decimal point, and forming the groups each way from the decimal point.

(Each group is called a period. The left-most period may contain only one figure. Thus, 62753.489 becomes 6 27 53.48 90.)

2. Find the largest square number not more than the left-most period. Write its square root as the first figure of the square root. Subtract the square number itself from the first period.

3. To the remainder, annex (bring down) the next period.

4. Form the trial divisor by doubling the root already found, and annexing a zero.

5. Divide the remainder formed in Step 3 by the trial divisor. Annex the quotient to the root already found, and add it to the trial divisor to form the complete divisor.

6. Multiply the complete divisor by the new figure of the root. Subtract the result from the remainder formed in Step 3.

7. If other periods remain, repeat Steps 4, 5, and 6 until there is not a remainder, or until the desired number of decimal places for the root have been obtained.

NOTE. — In Step 6, sometimes the product is greater than the remainder. In such cases, the last figure obtained for the root is too large. Substitute for it the next smaller integer.

Sometimes the next figure of the root must be made zero. In this case, bring down the next period and repeat Steps 4, 5, and 6.

Example 2. — Find the square root of 5207.0656.

Solution. — 1. The largest square number in 52 is 49. Write $\sqrt{49}$ or 7 in the root. Subtract. Annex 07.

2. Trial divisor is 2 × 7. Annex 0. 307 ÷ 140 = 2^+. Place 2 in the root. Add 2 to 140. Multiply 142 by 2. Subtract. Annex 06.

3. Trial divisor is 2 × 72. Annex 0. 2306 ÷ 1440 = 1^+. Place 1 in the root. Add 1 to 1440. Multiply 1441 by 1. Subtract. Annex 56.

4. Trial divisor is 2 × 721. Annex 0. 86556 ÷ 14420 = 6^+. Place 6 in the root. Add 6 to 14420. Multiply 14426 by 6. No remainder.

```
            7  2.1  6
         |52 07.06 56
         |49
  140    |3 07
    2    |
  142    |2 84
  1440   |  23 06
     1   |
  1441   |  14 41
  14420  |   8 65 56
      6  |
  14426  |   8 65 56
```

The square root is 72.16.

EXERCISE 120

Find the square root of:

1. 1296
2. 1849
3. 3721
4. 7396
5. 8836
6. 17,689
7. 33,489
8. 82,369
9. 96,100
10. 43,264
11. 164,025
12. 1474.56
13. 19.1844
14. 4160.25
15. 62.8849
16. 9.4249
17. 761.76
18. 5913.61
19. 8.5264
20. 64.6416

167. **The approximate square root of a number which is not a perfect square is often desired.**

Example. — Find $\sqrt{2}$ to three decimal places.

Solution. —

$$\begin{array}{r|l}
 & 1.\ 4\ \ 1\ \ 4 \\ \hline
 & 2.00\ 00\ 00 \\
 & 1 \\ \cline{2-2}
20 & 1.00 \\
\underline{4} & \\
24 & \ \ 96 \\ \cline{2-2}
280 & 4\ 00 \\
\underline{1} & \\
281 & 2\ 81 \\ \cline{2-2}
2820 & 1\ 19\ 00 \\
\underline{4} & \\
2824 & 1\ 12\ 96 \\ \cline{2-2}
 & \quad 6\ 04
\end{array}$$

Therefore the square root of 2 *to three decimal places* is 1.414. The root, *correct to hundredths*, is 1.41, since the figure in the third decimal place is less than 5. If it were more than 5, the figure in the hundredths place would be increased by 1. Thus, 3.146, *correct to hundredths*, is 3.15.

EXERCISE 121

Find the approximate square root to three decimal places:

1. 3
2. 5
3. 6
4. 7
5. 10
6. 11
7. 13
8. 14
9. 15
10. 17
11. 19
12. 21

168. **Table of square roots.** In the remainder of your courses in mathematics, you will need frequently the square roots of certain numbers. It is not economical to compute the same roots repeatedly. Therefore keep in a notebook any square roots which you compute. Begin now by writing the numbers from 1 to 50 in a column. Opposite each, in a second column, write the principal square root of the number to three decimal places.

After you have completed Exercise 121, you will have 12 of these square roots. Some others you can write down by inspection. Others can be got as follows.

Example. — $\sqrt{8} = \sqrt{4 \cdot 2} = 2\sqrt{2} = 2 \times (1.414) = 2.828.$

EXERCISE 122

Find each of the following, using any roots which you have already computed.

1. $\sqrt{12}$
2. $\sqrt{18}$
3. $\sqrt{20}$
4. $\sqrt{24}$
5. $\sqrt{27}$
6. $\sqrt{28}$
7. $\sqrt{45}$
8. $\sqrt{32}$
9. $\sqrt{40}$
10. $\sqrt{44}$

11. Complete the table of square roots to 50:
   a. getting as many as possible by inspection;
   b. getting others as in Examples 1–10 above;
   c. getting the rest by the process taught in § 166.

169. The square root of a fraction may be obtained in three ways.

Example 1. — Find $\sqrt{\frac{3}{2}}$ to three decimal places.

Method a. $\frac{3}{2} = 1.5.$ The square root can be found as in § 166. It cannot be got mentally.

Method b. $\sqrt{\frac{3}{2}} = \frac{\sqrt{3}}{\sqrt{2}} = \frac{1.732}{1.414} = ?$ The result must be found by long division.

Method c.
$$\sqrt{\frac{3}{2}} = \sqrt{\frac{6}{4}} = \frac{\sqrt{6}}{\sqrt{4}} = \frac{2.449}{2} = 1.224.$$

In methods *b* and *c*, the square roots are taken from your table of roots. If you do not have the necessary roots, then two roots must be computed in method *b*, and only one in method *c*.

Method *c* is clearly the one to use. Observe that the fraction $\frac{3}{2}$ is changed to an *equivalent fraction* (see page 169), whose denominator is the smallest possible perfect square.

Rule. — To find the square root of a fraction:

1. Change the fraction to an equivalent fraction whose denominator is the smallest possible perfect square.

2. The square root of the fraction equals the square root of its numerator divided by the square root of its denominator.

3. Express the result of Step 2 in simplest radical form, and also in decimal form, to three places.

Example 2. — Find $\sqrt{\frac{3}{8}}$.

Solution. — 1. The smallest perfect square into which 8 can be changed is 16. Multiply both terms of the fraction by 2.

2. $$\sqrt{\frac{3}{8}} = \sqrt{\frac{6}{16}} = \frac{\sqrt{6}}{\sqrt{16}} = \frac{2.449}{4} = .612.$$

EXERCISE 123

Find the square root of:

1. $\frac{2}{3}$

2. $\frac{3}{5}$

3. $\frac{5}{8}$

4. $\frac{2}{9}$

5. $\frac{3}{4}$

6. $\frac{5}{6}$

7. $\frac{2}{7}$

8. $\frac{8}{3}$

9. $\frac{9}{2}$

10. $\frac{7}{8}$

11. $\frac{3}{7}$

12. $\frac{9}{10}$

13. $\frac{4}{11}$

14. $\frac{5}{12}$

15. $\frac{9}{13}$

170. A **quadratic surd** is the indicated square root of a number which is not a perfect square; as $\sqrt{3}$. **The square root symbol indicates the principal square root.**

A surd is in its simplest form when the number under the radical sign is an integer which does not have any perfect square factors.

Thus, $\sqrt{24}$, simplified, becomes $\sqrt{4 \cdot 6}$ or $2\sqrt{6}$.
$\sqrt{\frac{5}{8}}$, simplified, becomes $\sqrt{\frac{10}{16}}$ or $\frac{1}{4}\sqrt{10}$.

171. A **surd expression** is an expression involving one or more surds. In this chapter, only quadratic surd expressions are considered.

A surd expression is simplified by first simplifying the surds, and then performing all other indicated operations.

Example. — Simplify $\frac{3}{7} - \sqrt{\frac{2}{7}}$.

Solution. — 1. $$\frac{3}{7} - \sqrt{\frac{2}{7}} = \frac{3}{7} - \sqrt{\frac{14}{49}}$$

2. $$= \frac{3}{7} - \frac{\sqrt{14}}{7}$$

3. $$= \frac{3 - \sqrt{14}}{7}.$$

Find the value of the expression to three decimal places.

Solution. — $$\frac{3 - \sqrt{14}}{7} = \frac{3 - 3.741}{7} = \frac{-.741}{7} = -.105.$$

EXERCISE 124

a. Simplify the following expressions; *b.* express the results to three decimal places:

1. $6 + \sqrt{12}$
2. $-5 + \sqrt{18}$
3. $4 - \sqrt{20}$
4. $-3 - \sqrt{8}$
5. $+7 + \sqrt{24}$
6. $\frac{1}{2} + \sqrt{\frac{1}{2}}$
7. $\frac{1}{2} - \sqrt{\frac{1}{2}}$
8. $\frac{3}{4} + \sqrt{\frac{1}{8}}$
9. $\frac{3}{4} - \sqrt{\frac{5}{8}}$
10. $\frac{2}{3} + \sqrt{\frac{5}{3}}$
11. $-\frac{2}{3} - \sqrt{\frac{5}{3}}$
12. $-\frac{2}{5} + \sqrt{\frac{3}{5}}$
13. $-\frac{2}{5} - \sqrt{\frac{2}{5}}$
14. $+\frac{5}{6} - \sqrt{\frac{5}{18}}$
15. $-\frac{5}{6} + \sqrt{\frac{5}{18}}$

SUPPLEMENTARY TOPICS

172. Further addition and subtraction of surds.

Example 1. — Find $\sqrt{20} + \sqrt{45}$.

Solution. — 1. $\sqrt{20} + \sqrt{45} = \sqrt{4 \cdot 5} + \sqrt{9 \cdot 5}$

2. $= 2\sqrt{5} + 3\sqrt{5}$

3. $= 5\sqrt{5}$.

This is the sum in its simplest radical form. In this form, its value to any desired number of decimal places is found with a minimum of computation.

$$5\sqrt{5} = 5 \times 2.236 = 11.180.$$

The surds expressions $2\sqrt{5}$ and $3\sqrt{5}$ are called *similar surds,* because the surd factor of each is the same. *Only similar surds can be combined.*

Example 2. — Find the simplest radical form and also the decimal value to three places of $\sqrt{\frac{9}{8}} + \sqrt{\frac{1}{2}}$.

Solution. — 1. $\sqrt{\frac{9}{8}} + \sqrt{\frac{1}{2}} = \sqrt{\frac{9 \cdot 2}{16}} + \sqrt{\frac{2}{4}}$

2. $= \frac{3\sqrt{2}}{4} + \frac{\sqrt{2}}{2}$

3. $= \frac{3\sqrt{2} + 2\sqrt{2}}{4}$, or $\frac{5\sqrt{2}}{4}$.

4. $\frac{5\sqrt{2}}{4} = \frac{5 \times 1.414}{4} = \frac{7.07}{4} = 1.767.$

EXERCISE 125

Find the simplest radical form and also the decimal value to three places of the following expressions:

1. $\sqrt{12} + \sqrt{27}$
2. $\sqrt{75} - \sqrt{48}$
3. $\sqrt{18} + \sqrt{32}$
4. $\sqrt{45} - \sqrt{5}$
5. $3\sqrt{6} - \sqrt{24}$
6. $\sqrt{28} - 3\sqrt{7} - \sqrt{63}$
7. $\sqrt{50} - \sqrt{18} + \sqrt{8}$
8. $\sqrt{80} - \sqrt{45} + \sqrt{5}$
9. $\sqrt{48} - 2\sqrt{3} + \sqrt{300}$
10. $\sqrt{200} - \sqrt{98} - \sqrt{72}$
11. $\sqrt{\frac{1}{2}} + \sqrt{8}$
12. $\sqrt{\frac{3}{4}} + \sqrt{\frac{4}{3}}$
13. $\sqrt{\frac{2}{3}} - \sqrt{\frac{1}{6}}$
14. $\sqrt{\frac{2}{5}} + \sqrt{40}$
15. $\sqrt{\frac{3}{8}} - \sqrt{\frac{2}{3}}$
16. $\sqrt{\frac{2}{9}} + \sqrt{\frac{1}{8}} - \sqrt{\frac{1}{2}}$
17. $\sqrt{\frac{4}{3}} - \sqrt{\frac{1}{27}} + \sqrt{\frac{1}{12}}$
18. $\sqrt{\frac{1}{10}} + \sqrt{90} - \sqrt{4\frac{9}{10}}$
19. $\sqrt{\frac{1}{6}} + \sqrt{\frac{2}{3}} - \sqrt{\frac{3}{8}}$
20. $\sqrt{\frac{1}{5}} + \sqrt{20} - \sqrt{\frac{45}{4}}$

173. Multiplication of quadratic surds.

The fact that $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$ has been used repeatedly. Thus,

$\sqrt{4 \cdot 9} = \sqrt{4} \cdot \sqrt{9} = 2 \cdot 3 = 6$; also $\sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}$.

Since $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$, then also $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$.

Thus, $\sqrt{2} \cdot \sqrt{3}$ must equal $\sqrt{2 \cdot 3}$, or $\sqrt{6}$.

Rule. — The product of the square roots of two numbers equals the square root of the product of the numbers.

Example. — Find the simplest radical form and also the decimal value to three places of $2\sqrt{3} \times 5\sqrt{2}$.

Solution. — 1. $2\sqrt{3} \times 5\sqrt{2} = 10\sqrt{3 \cdot 2} = 10\sqrt{6}$.

2. $10\sqrt{6} = 10 \times 2.449 = 24.49$.

When doing such multiplication, keep the product under the radical sign in factored form.

Thus, $\sqrt{18} \cdot \sqrt{12} = \sqrt{18 \cdot 12} = \sqrt{9 \cdot 2 \cdot 4 \cdot 3} = 3 \cdot 2\sqrt{6}$, or $6\sqrt{6}$.

EXERCISE 126

Find the simplest radical form and also the decimal value to three places of:

1. $\sqrt{2} \cdot \sqrt{8}$
2. $\sqrt{3} \cdot \sqrt{12}$
3. $\sqrt{5} \cdot \sqrt{15}$
4. $\sqrt{6} \cdot \sqrt{12}$
5. $\sqrt{7} \cdot \sqrt{21}$
6. $\sqrt{11} \cdot \sqrt{22}$
7. $3\sqrt{2} \cdot 4\sqrt{5}$
8. $2\sqrt{6} \cdot 5\sqrt{3}$
9. $6\sqrt{20} \cdot 3\sqrt{10}$
10. $(3\sqrt{3})^2$
11. $(2\sqrt{2})^2$
12. $(\frac{1}{3}\sqrt{2})^2$

13. Multiply $3 + 5\sqrt{2}$ by $3 + 5\sqrt{2}$.

Solution. —

$$\begin{array}{l} 3 + 5\sqrt{2} \\ \underline{3 + 5\sqrt{2}} \\ 9 + 15\sqrt{2} \\ \underline{\qquad 15\sqrt{2} + 25\sqrt{4}} \\ 9 + 30\sqrt{2} + 50 \qquad \text{or} \quad 59 + 30\sqrt{2} \end{array}$$

The decimal value of the product is $59 + 30 \times 1.414$, or 101.42.

Find:

14. $(2 - \sqrt{3})(2 - \sqrt{3})$
15. $(5 + \sqrt{6})(5 + \sqrt{6})$
16. $(-3 + \sqrt{2})(-3 + \sqrt{2})$
17. $(9 - \sqrt{5})(9 - \sqrt{5})$
18. $(-7 + \sqrt{11})(-7 + \sqrt{11})$
19. $(10 - \sqrt{6})(10 + \sqrt{6})$
20. $(3 - 2\sqrt{2})(3 - 2\sqrt{2})$
21. $(5 + 3\sqrt{7})(5 + 3\sqrt{7})$
22. $(-4 + 5\sqrt{3})^2$
23. $(6 - 2\sqrt{7})^2$
24. $(-3 + 5\sqrt{6})^2$
25. $(11 - 3\sqrt{5})^2$

XIV. QUADRATIC EQUATIONS

174. A **pure quadratic equation** is an equation in one unknown in which the unknown does not appear in the denominator of a fraction, and in which only the second power of the unknown does appear.

Thus, $5x^2 - 37 = 0$ is a pure quadratic equation.

Example. — How long must the side of a square field be in order that the area of the field shall be one acre (43,560 sq. ft.)?

Solution. — 1. Let s = the no. of ft. in one side of the square.

2. $\therefore s^2 =$ the area of the square.

3. $\therefore s^2 = 43{,}560.$

4. Take the square root of both members of the equation.

$$\therefore s = \sqrt{43{,}560}.$$

5. $\therefore s = \pm 208.7$ ft.

6. Since s is the length of a side of the field, only the positive root has meaning. That is, the side must be 208.7^+ ft.

175. A pure quadratic equation has two roots.

Example. — $x^2 = 16.$

Solution. — When we take the square root of both sides, really we get $\pm x = \pm 4$ This means:

(1) $+x = +4$ (3) $-x = +4$

(2) $+x = -4$ (4) $-x = -4$

However, if both members of equation (3) are multiplied by -1, we get $+x = -4$, which is equation (2); and if both members of equation (4) are multiplied by -1, we get $+x = +4$, which is equation (1).

That is, we actually get only two equations, and not four.

We get these two equations, moreover, if we use only the positive root on the left side and the two roots on the right side; that is, $+x = \pm 4$.

Rule. — To solve a pure quadratic equation:

1. Simplify the equation until it takes the form x^2 = a number.

2. Take the square root of both members of the equation, placing the + sign in front of x, and the double sign (±) before the square root of the right member.

Example. — Solve the equation $\frac{2m}{3} + \frac{3}{m} = \frac{m}{12} + \frac{12}{m}$.

Solution. — 1. $\frac{2m}{3} + \frac{3}{m} = \frac{m}{12} + \frac{12}{m}$.

2. M_{12m} $8m^2 + 36 = m^2 + 144$.

3. Simplifying, $7m^2 = 108$.

4. M_7 $m^2 = \frac{108}{7}$.

5. $m = \pm\sqrt{\frac{108}{7}} = \pm 6\sqrt{\frac{3}{7}} = \pm \frac{6}{7}\sqrt{21}$.

6. $m = \pm\frac{6}{7}\cdot(4.582) = \pm\frac{27.492}{7} = \pm 3.927$.

7. $m_1 = +3.927$; $m_2 = -3.927$.

"m_1" is read "m one." The numeral 1 is called in such cases a *subscript.* "m_2" is read "m two." These subscripts are used in this solution to distinguish between the two roots of the quadratic.

Check. — In cases such as this, it is better to check by going over the solution a second time. Great care must be taken, however, for it is easy to overlook an error.

EXERCISE 127

1. $11x^2 - 99 = 0$
2. $5x^2 - 125 = 3x^2 - 27$
3. $13c^2 + 8 = 88 - 7c^2$
4. $5(m - 15) + m(3m - 5) = 0$
5. $7(t - 2) + t(t - 5) = 2t$
6. $(m + 3)(m - 1) = m(2 - 3m)$
7. $(c - 2)^2 = (2c - 8)(c + 2)$
8. $(3m - 2)(3m + 1) = 3(3 - m)$
9. $(2x - 3)^2 = 2(x - 5)(x - 1)$
10. $(5c + 1)(c - 3) - 2(2 - 7c) = 0$

11. $\frac{1}{2s} - \frac{1}{3s} = \frac{s}{4}$

12. $\frac{4r - 1}{2r + 1} = \frac{r + 2}{3r + 2}$

13. $\frac{x + 3}{x - 3} + \frac{x - 3}{x + 3} = 3$

14. $\frac{4y^2 + 5}{3} - \frac{3y^2 + 1}{6} = \frac{5y^2 + 15}{9}$

15. $\frac{2}{r + 1} + \frac{3}{2r - 1} = 7$

16. $\frac{x^2 + x + 1}{x - 1} = \frac{x^2 + 5x + 7}{x + 1}$

17. Solve for x: $a - 2cx^2 = 3b$.

Solution.—1. $a - 2cx^2 = 3b$.

2. S_a $-2cx^2 = 3b - a$.

3. D_{-2c} $x^2 = \frac{a - 3b}{2c}$.

4. $x = \pm\sqrt{\frac{a - 3b}{2c}} = \pm\sqrt{\frac{2c(a - 3b)}{4c^2}}$.

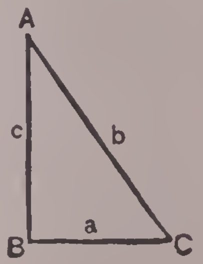
5. $\therefore x = \pm\frac{1}{2c}\sqrt{2c(a - 3b)}$.

Solve for x:

18. $a^2 + x^2 = m$ **19.** $cx^2 = d$ **20.** $a + bx^2 = c$

21. Solve $s = \frac{1}{2}gt^2$ for t. **22.** Solve $f = \frac{mv^2}{r}$ for v.

176. A **right triangle** is a triangle which has a right angle for one of its angles; as triangle ABC, in which angle B is a right angle.



The side opposite the right angle is the *hypotenuse*; as, side AC. The side BC is the *base* and AB is the *altitude.*

In a right triangle, the square of the hypotenuse equals the sum of the squares of the other two sides.

Thus, $b^2 = a^2 + c^2$.

To verify this fact, draw a right triangle with BC 3 inches and AB 4 inches; measure AC. Substitute the lengths of the sides in the equation $b^2 = a^2 + c^2$.

EXERCISE 128

Carry out all computations in this set to two decimal places.

1. Find the altitude of a right triangle whose base is 15 feet and whose hypotenuse is 30 feet.

Solution. — 1. The formula is $b^2 = a^2 + c^2$.

2. In this problem $a = 15$, $b = 30$, and $c = ?$

3. $\therefore\ 30^2 = 15^2 + c^2$.

(Complete the solution.)

2. Find the base of the right triangle whose hypotenuse is 35 feet and whose altitude is 20 feet.

3. If the diagonal of a rectangle is 65 inches and the base of it is 3 times the altitude, what are the dimensions of the rectangle?

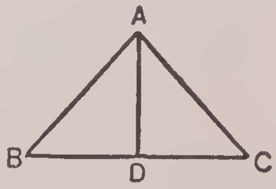
4. *a.* The altitude of a rectangle is a inches. The base is 3 times as long. What is the length of the diagonal?

b. Using the result of part *a* as a formula, how long is the diagonal when $a = 5$? $a = 8$?

5. Solve the formula $b^2 = a^2 + c^2$: *a.* for b; *b.* for c.

c. Express the results of parts *a* and *b* in words.

6. Triangle ABC at the right is an **isosceles triangle.** AD is perpendicular to BC; it is the **altitude** to the base BC.



a. Compare BD and DC by measurement.

b. Suppose $AB = 15$ inches and BC is 20 inches. Find AD.

7. If the equal sides of an isosceles triangle are each 25 inches long, and the altitude is 15 inches, find the base.

8. *a.* If the equal sides of an isosceles triangle are each m inches long and the base is $2n$ inches long, derive the formula for the altitude.

b. Using your formula, find the altitude when the sides measure 20 inches and the base 14 inches.

9. An **equilateral triangle** has all three sides equal. What is the altitude of the equilateral triangle whose sides are 12 inches long?

10. Find the altitude of the equilateral triangle whose sides are s units long.

11. Recall the formula for the area of a circle. (See page 29.) What is the area when the radius is 8 inches?

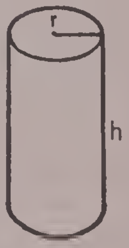
12. What is the radius of the circle which incloses an acre of ground (43,560 sq. ft.)?

13. Solve the formula $A = \pi r^2$ for r, in simplest radical form:

a. letting $\pi = 3\frac{1}{7}$; *b.* without substituting a value for π.

14. The volume of a circular cylinder is given by the formula $V = \pi r^2 h$, where r = the radius, and h = the altitude.

Find V when $r = 7$ and $h = 20$.



15. Find r when $V = 1100$ and $h = 14$.

16. Solve the formula of Example 14 for r.

17. The formula for the area of the surface of a sphere is $S = 4\pi r^2$.

Find S when $r = 15$ in.

18. Find in inches the radius of the sphere whose surface area is 1 square yard.

19. Solve the formula $S = 4\pi r^2$ for r, without substituting its value for π.

20. Solve the formula $V = \frac{1}{3}\pi r^2 h$ for r.

177. A complete quadratic equation is an equation having only one unknown, in which the unknown does not appear in the denominator of the equation, and in which the first and second powers of the unknown do appear.

Thus, $2x^2 - 3x - 5 = 0$ is a complete quadratic equation.

178. Solution by factoring was taught on pages 148 and 149.

EXERCISE 129

Solve for the unknown number:

1. $y^2 - 6y = 55$
2. $8a^2 - 5a - 3 = 0$
3. $3w^2 - \frac{7}{2}w + 1 = 0$
4. $\frac{3}{10}m^2 = 1 - \frac{13}{10}m$

Solve for x:

5. $7x^2 - 10bx + 3b^2 = 0$
6. $\frac{x^2}{6} + 6c^2 = \frac{5cx}{2}$
7. $6x^2 = \frac{47mx}{5} + m^2$
8. $\frac{3x^2}{10} - \frac{7xt}{5} = \frac{t^2}{2}$
9. $\frac{x^2}{10} - \frac{rx}{6} = \frac{11r^2}{15}$
10. $\frac{x^2}{5} - \frac{2xs}{15} = \frac{s^2}{4}$

Solve for the unknown number:

11. $\frac{5 - x}{1 - x} = \frac{9}{5 - 4x}$
12. $\frac{8}{y + 1} = \frac{12}{y - 2} - 5$
13. $\frac{9}{z + 3} - \frac{3}{2z - 3} = \frac{1}{2}$
14. $\frac{1}{1 - 2t} + \frac{1}{1 - 4t} = \frac{5}{6}$
15. $\frac{r + 1}{r - 1} = \frac{2r - 1}{4r + 1}$
16. $\frac{2s - 5}{3s + 5} - \frac{s + 1}{5s - 1} = 0$

179. Solution of complete quadratics by completing the square.

Preparation. — 1. Find: *a.* $(x + 4)^2$; *b.* $(y - 5)^2$; *c.* $(z + \frac{1}{3})^2$; *d.* $(w - \frac{1}{4})^2$; *e.* $(r - \frac{3}{5})^2$; *f.* $(s - \frac{2}{7})^2$.

2. Make a perfect square trinomial of each of the following expressions (see page 249), and give its square root.

a. $x^2 + 8x$; *b.* $y^2 - 14y$; *c.* $w^2 - 20w$; *d.* $m^2 - \frac{2}{3}m$.

3. Notice that the coefficient of the second power in each of these expressions is 1; that the term added in each case is the square of one half the coefficient of the first power; thus, in *a*, you add $(\frac{8}{2})^2$.

This is called completing the square. It is used in solving quadratics.

Example 1. — Solve $x^2 - 12x + 20 = 0$ by completing the square.

Solution. — 1. S_{20} $x^2 - 12x = -20$.

2. We can complete the square of the left member by adding $(\frac{12}{2})^2$ or 36. But then we must also add 36 to the right member.

3. A_{36} $x^2 - 12x + 36 = -20 + 36$.
4. $\therefore (x - 6)^2 = 16$.
5. $\surd$ $x - 6 = \pm 4$. Read Note, page 260.
6. $\therefore x_1 - 6 = +4$, or $x_1 = 10$. This is one root.
7. $\therefore x_2 - 6 = -4$, or $x_2 = 2$. This is the other root.

NOTE 1. — For meaning of x_1 and x_2, see Example, page 261.

Check. — *a.* These roots check when substituted in equation 1.

b. A second convenient method of checking.

Add the roots: $10 + 2 = 12$. *Observe that this is the negative of the coefficient of x in the original equation.*

Multiply the roots: $10 \cdot 2 = 20$. *Observe that this is the term free from x in the original equation.*

This method of checking may be used only when the coefficient of the second power is 1.

Example 2. — Solve $x^2 - 6x - 5 = 0$.

Solution. — 1. A_5 $x^2 - 6x = 5$.
2. $(-6) \div 2 = -3$; $(-3)^2 = +9$.
3. A_9 $x^2 - 6x + 9 = 5 + 9$.
4. $\therefore (x - 3)^2 = 14$.
5. $x - 3 = \pm\sqrt{14}$.
6. When $x - 3 = +\sqrt{14}$, $x_1 = 3 + \sqrt{14}$. These are the roots
When $x - 3 = -\sqrt{14}$, $x_2 = 3 - \sqrt{14}$. in radical form.

7. $\therefore x_1 = 3 + \sqrt{14} = 3 + 3.741 = 6.741.$ These are the roots
$\therefore x_2 = 3 - \sqrt{14} = 3 - 3.741 = -.741.$ in decimal form.

Check. — Method *a*. When $x = 6.741$,

does $(6.741)^2 - 6(6.741) - 5 = 0?$

Does $45.441 - 40.446 - 5 = 0?$

Does $45.441 - 45.446 = 0?$ Almost.

Since the roots are correct only as far as the third decimal place, *we do not expect this root to check exactly.*

Method *b*. Adding the roots

$$\begin{array}{r} 6.741 \\ -\ .741 \\ \hline +\ 6. \end{array}$$

This gives $-(-6)$ as it should.

Multiply the roots:

$$\begin{array}{r} 6.741 \\ -\ .741 \\ \hline 6\ 741 \\ 269\ 64 \\ 4\ 718\ 7 \\ \hline -\ 4.995\ 081 \end{array}$$

This gives -4.99^+. It should give -5. The result is near enough to indicate that the roots are correct.

The only roots which will check absolutely are the radical roots.

Rule. — To solve a quadratic equation by completing the square:

1. Simplify the equation; transpose all terms containing the unknown number to the left member, and all other terms to the right member so that the equation takes the form

$$ax^2 + bx = c.$$

2. If the coefficient of x^2 is not 1, divide both members of the equation by it, so that the equation takes the form

$$x^2 + px = q.$$

3. Find one half of the coefficient of x; square the result; add the square to both members of the equation obtained in Step 2. This makes the left member a perfect square.

4. Write the left member as the square of a binomial; express the right member in its simplest form.

5. Take the square root of both members, writing the double sign, ±, before the square root in the right member.

6. Set the left square root equal to the + root in the right member of the equation in Step **5**. Solve for the unknown. This gives one root.

7. Repeat the process, using the − root in Step **5**. This gives the second root of the equation.

8. Express the roots first in simplest radical form, and then in simplest decimal form.

EXERCISE 130

Solve by completing the square. Check 1–6. Check others as directed by your teacher.

1. $x^2 - 8x + 15 = 0$ **5.** $y^2 - 4y + 1 = 0$
2. $y^2 + 8y + 12 = 0$ **6.** $z^2 = 6z - 4$
3. $z^2 + 2z - 35 = 0$ **7.** $w^2 + 10w + 22 = 0$
4. $x^2 - 2x - 1 = 0$ **8.** $s^2 + 8s + 10 = 0$

9. Solve the equation $x^2 - 9x + 7 = 0$.

Solution.—1. S_7 $x^2 - 9x = -7.$
2. $(-9) \div 2 = -\frac{9}{2}; \quad (-\frac{9}{2})^2 = \frac{81}{4}.$
3. $\text{A}_{\frac{81}{4}}$ $x^2 - 9x + \frac{81}{4} = \frac{81}{4} - 7.$
4. $\therefore (x - \frac{9}{2})^2 = \frac{53}{4}.$
5. $x - \frac{9}{2} = \pm \frac{\sqrt{53}}{2}.$
6. $x = \frac{9}{2} \pm \frac{\sqrt{53}}{2}.$
7. $x = \frac{9 \pm \sqrt{53}}{2}.$
8. Then $x_1 = \frac{9 + \sqrt{53}}{2} = \frac{9 + 7.280}{2} = \frac{16.28}{2} = 8.14$

and $x_2 = \frac{9 - \sqrt{53}}{2} = \frac{9 - 7.280}{2} = \frac{1.72}{2} = .86.$

(Check this solution in one of the two ways taught.)

10. $x^2 + 3x - 4 = 0$ **13.** $m^2 + m = 5$
11. $y^2 + 5y = 14$ **14.** $c^2 - 7c + 3 = 0$
12. $z^2 - 3z + 1 = 0$ **15.** $r^2 - 9r = 12$

16. $t^2 - 11\,t + 6 = 0$

17. $a^2 - 15\,a + 56 = 0$

18. $s^2 - 17\,s + 10 = 0$

19. $x^2 - 21\,x - 22 = 0$

20. Solve Example 19 by the factoring method.

21. Solve the equation $x^2 + \frac{2}{5}\,x = \frac{3}{5}$.

Solution. — 1. $x^2 + \frac{2}{5}\,x = \frac{3}{5}$.

2. $\frac{1}{2}$ of $\frac{2}{5} = \frac{1}{5}$; $(\frac{1}{5})^2 = \frac{1}{25}$.

3. $x^2 + \frac{2}{5}\,x + \frac{1}{25} = \frac{3}{5} + \frac{1}{25}$.

4. $(x + \frac{1}{5})^2 = \frac{16}{25}$.

(Complete the solution.)

22. $x^2 + \frac{4}{3}\,x + \frac{1}{3} = 0$

23. $x^2 - \frac{4}{5}\,x - \frac{3}{5} = 0$

24. $x^2 - \frac{8}{3}\,x - 2 = 0$

25. $x^2 + \frac{1}{2}\,x = 3$

26. $x^2 - \frac{3}{4}\,x = 2$

27. $y^2 - \frac{3}{5}\,y - 3 = 0$

28. Solve the equation $3\,x^2 - 5\,x + 2 = 0$.

Solution. — 1. $3\,x^2 - 5\,x + 2 = 0$.

2. In order to make the coefficient of x^2 be 1,

D_3 $x^2 - \frac{5}{3}\,x + \frac{2}{3} = 0$.

(Complete the solution.)

29. $2\,x^2 + 5\,x + 2 = 0$

30. $3\,x^2 - 10\,x + 3 = 0$

31. $6\,x^2 - x - 2 = 0$

32. $4\,p^2 - 4\,p = 1$

33. $5\,r^2 = 2\,r + 7$

34. $3\,m^2 - 5\,m - 4 = 0$

35. $7\,x^2 - 2\,x - 3 = 0$

36. $8\,y^2 - 4\,y - 1 = 0$

37. $9\,w^2 - 6\,w - 4 = 0$

38. $6\,t^2 - 8\,t - 5 = 0$

39. $12\,s^2 - 10\,s - 9 = 0$

40. $10\,w^2 - 4\,w - 3 = 0$

180. Solution of literal quadratics by completing the square.

Example. — Solve for x the equation $x^2 + 3\,ax + 4\,b = 0$.

Solution. — 1. $x^2 + 3\,ax = -4\,b$.

2. $\frac{1}{2}$ of $3\,a = \left(\frac{3\,a}{2}\right)$. $\left(\frac{3\,a}{2}\right)^2 = \frac{9\,a^2}{4}$.

3. $A_{\left(\frac{9a^2}{4}\right)}$ $x^2 + 3\,ax + \frac{9\,a^2}{4} = \frac{9\,a^2}{4} - 4\,b$.

4. $$\left(x + \frac{3a}{2}\right)^2 = \frac{9a^2 - 16b}{4}.$$

5. $$\therefore\ x + \frac{3a}{2} = \pm\frac{\sqrt{9a^2 - 16b}}{2}.$$

6. $$x = \frac{-3a}{2} \pm \frac{\sqrt{9a^2 - 16b}}{2}$$

7. $$= \frac{-3a \pm \sqrt{9a^2 - 16b}}{2}.$$

8. $$x_1 = \frac{-3a + \sqrt{9a^2 - 16b}}{2};\quad x_2 = \frac{-3a - \sqrt{9a^2 - 16b}}{2}.$$

NOTE 1.—This solution is now complete. $9a^2 - 16b$ is not a perfect square and does not contain any square factors.

NOTE 2.—The solution can be checked by each of the methods proposed on page 266. Probably it is as satisfactory to check this solution by going over it carefully.

EXERCISE 131

Solve the following equations for x:

1. $x^2 + 2ax + 3 = 0$
2. $x^2 + 4ax + b = 0$
3. $x^2 - 6bx - c = 0$
4. $4x^2 + 8x + m = 0$
5. $2x^2 + x - n = 0$
6. $3x^2 + 2ax + b = 0$
7. $3x^2 + ax + b = 0$
8. $x^2 - 2mx = 1 - 2m$
9. $x^2 + 6x = t^2 + 6t$
10. $x^2 - 4ax = b^2 - 4ab$
11. $ax^2 + 4x + 1 = 0$
12. $ax^2 + x + 1 = 0$
13. $ax^2 + 2dx + m = 0$
14. $ax^2 + bx + c = 0$

181. Solution of quadratic equations by a formula. All quadratic equations having one unknown may be put in the form

$$ax^2 + bx + c = 0.$$

For example, $2x^2 - 3x - 5 = 0$ has this form. In it,

$$a = 2,\ b = -3, \text{ and } c = -5.$$

If we solve $ax^2 + bx + c = 0$ for x, the roots may be used as formulæ for the roots of any quadratic equation.

Derivation of the formula:

1. $$ax^2 + bx + c = 0.$$

2. D_a $$x^2 + \frac{b}{a} \cdot x + \frac{c}{a} = 0.$$

3. $S_{\frac{c}{a}}$ $$x^2 + \frac{b}{a} \cdot x = -\frac{c}{a}.$$

4. $$\frac{1}{2} \text{ of } \left(\frac{b}{a}\right) = \frac{b}{2a}; \quad \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}.$$

5. $A_{\left(\frac{b^2}{4a^2}\right)}$ $$x^2 + \frac{b}{a} \cdot x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}.$$

6. $$\therefore \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

7. $$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

8. $$\therefore x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

9. $$\therefore \boxed{\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}}$$

Read this solution carefully and be sure you understand it. But above all, **memorize the formula.** You can do so in two minutes.

Example showing the use of this important formula.

Solve the equation $2x^2 - 3x - 5 = 0$, by the formula.

Solution. — 1. Comparing this equation with $ax^2 + bx + c = 0$

$$a = 2, \; b = -3, \; c = -5.$$

2. The formula is $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

3. Substituting, $$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-5)}}{2 \cdot 2}.$$

4. $$\therefore x = \frac{+3 \pm \sqrt{9 + 40}}{4}.$$

5. $$\therefore x = \frac{+3 \pm \sqrt{49}}{4} \text{ or } \frac{+3 \pm 7}{4}.$$

6. $$\therefore x_1 = \frac{3 + 7}{4} = \frac{10}{4} = \frac{5}{2}; \text{ and } x_2 = \frac{3 - 7}{4} = \frac{-4}{4} = -1.$$

Check. — These roots check when substituted in the given equation.

EXERCISE 132

1. $x^2 - 7x + 6 = 0$

Hint. — $a = 1$; $b = -7$; $c = 6$.

2. $x^2 + 2x - 15 = 0$
3. $2x^2 + 5x + 2 = 0$
4. $3y^2 - 7y + 2 = 0$
5. $2z^2 + 7z + 3 = 0$
6. $2w^2 + 3w - 2 = 0$
7. $x^2 - 2x - 35 = 0$
8. $12x^2 + 5x - 2 = 0$
9. $10m^2 - 11m - 6 = 0$
10. $24x^2 + 14x - 3 = 0$
11. $x^2 - 2x - 2 = 0$
12. $x^2 + 2x - 1 = 0$
13. $2x^2 + 3x - 1 = 0$
14. $3y^2 - 4y - 2 = 0$
15. $2z^2 - 8z + 3 = 0$
16. $4t^2 - 9t + 2 = 0$
17. $5c^2 - 11c - 2 = 0$
18. $6m^2 = 5m + 3$
19. $7a^2 = 4 - 6a$
20. $8x^2 + 5x - 1 = 0$
21. $5x^2 - 3x - 1 = 0$

182. Three methods of solving a quadratic equation have been taught:

a. by factoring, in § 178, page 265.
b. by completing the square, in § 179, page 266.
c. by the formula, in § 181, page 271.

Always try to use the factoring method first.

If the expression is not easily factored, use one of the other two methods, preferably the third.

Historical Note. — Greek mathematicians as early as Euclid were able to solve certain quadratics by a geometric method, about which the student may learn when he studies plane geometry. Heron of Alexandria, about 110 B.C., proposed a problem which leads to a quadratic. His solution is not given, but his result would indicate that he probably solved the equation by a rule which might be obtained from the quadratic by completing its square in a certain manner. Diophantus, 275 A.D., gave many problems which lead to quadratic equations. The rules by which he solved his equations appear to have been derived by completing the square. He considered three separate kinds of quadratics. He gave only one root for a quadratic.

The Hindu mathematicians, knowing about negative numbers, considered one general quadratic. Cridharra gave a rule much like our formula. The Hindus knew that a quadratic has two roots, but they usually rejected any negative roots.

The Arabians went back to the practice of Diophantus in considering three or more kinds of quadratics. Mohammed Ben Musa, 820 A.D., had five kinds. He admitted two roots when both were positive. Alkarchi gave a purely algebraic solution of a quadratic by completing the square, and refers to this method as being a diophantic method.

In Europe, mathematicians followed the practice of the Arabians, and by the time of Widmann, 1489, had twenty-four special forms of equations. These were solved by rules which were learned and used in a mechanical manner. Stifel, 1486–1567, finally brought the study of quadratics back to the point that had been reached by the Hindus one thousand years before. He gave only three normal forms for the quadratic; he allowed double roots when they were both positive. Stevin, 1548–1620, went still farther. He gave only one normal form for the general quadratic, as do we; he solved this in both a geometric and an algebraic manner, giving the method of completing the square. He allowed negative roots.

EXERCISE 133

Solve the following equations by factoring if you can; otherwise by one of the other methods as your teacher directs.

1. $\frac{8}{x} - \frac{5}{x+3} = 3$

2. $\frac{3}{x+4} - \frac{4}{x-1} = 4$

3. $\frac{2}{x-1} - \frac{1}{3x+1} = \frac{1}{4}$

4. $\frac{2}{y+4} - \frac{1}{y-2} = 2$

5. $t - \frac{6}{t-3} = 0$

6. $\frac{3}{r+1} + \frac{4}{r} = -2$

7. $\frac{s}{s-1} - \frac{s}{2} = 1$

8. $\frac{2c}{c+5} - \frac{4}{c-3} = 1$

9. $\frac{3m}{2m+1} - \frac{2m}{3m-1} = 1$

10. $\frac{2}{4y+1} - \frac{1}{5} = \frac{3}{2y-1}$

11. $\frac{3x-1}{x+1} - \frac{2x+1}{x-1} = 3$

12. $\frac{3w-2}{5} - \frac{w-3}{w+3} = 1$

13. $\frac{7-3m}{2} - \frac{3+m}{2-m} = 5$

14. $\frac{2c-1}{3c-5} = \frac{3c+1}{4c} - \frac{1}{6}$

15. $\dfrac{5t+1}{3}+\dfrac{t-2}{t-3}=1$

16. $\dfrac{3z+5}{3z+2}+\dfrac{2z+1}{2z}=1$

17. $\dfrac{2a-3}{a+3}+\dfrac{a-3}{2a+3}=1$

18. $\dfrac{3r-1}{5+r}-\dfrac{2-r}{2r+1}=2$

19. $\dfrac{5c}{2c-5}-\dfrac{3}{c-4}=2$

20. $\dfrac{6-5m}{2}-\dfrac{3+m}{2-m}=\dfrac{3}{2}$

183. Problems solved by quadratics. Review § 49, page 58, before solving the following problems.

EXERCISE 134

1. If 3 be subtracted from 8 times a certain number, the result is 4 times the square of the number. What is the number?

2. If 7 times the square of a certain number be increased by 3, the result equals 10 times the number. What is the number?

3. There are two consecutive integers whose product is 462. What are they?

4. The sum of the squares of certain two consecutive integers is 481. What are they?

5. There are three consecutive even integers such that the square of the first, increased by the product of the other two, is 184. What are they?

6. The sum of a certain number and its reciprocal is $\frac{29}{10}$. What is the number?

Hint. — Read page 245.

7. The length of a certain rectangle exceeds its width by 8 feet. Its area is 308 square feet. What are its dimensions?

8. The denominator of a certain fraction exceeds its numerator by 5. The sum of the fraction and its reciprocal is $\frac{73}{24}$. What is the fraction?

9. The base of a certain triangle exceeds its altitude by 5 inches. Its area is 88 square inches. What are its dimensions?

10. The base of a certain rectangle is 9 feet longer than its altitude. Its area equals that of a square whose side is 20 feet. What are its dimensions?

11. The area of a certain rectangle is 660 square feet. The sum of its base and altitude is 52 feet. What are its dimensions?

12. The total area of certain two squares is 369 square feet. The side of one exceeds the side of the other by 3 feet. How long is the side of each?

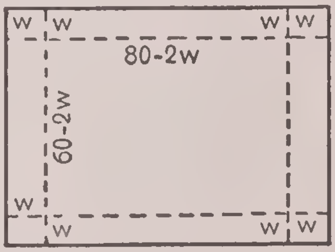
13. The area of the main waiting room of the Union Railway Station at Washington, D. C., is 28,600 square feet. The sum of its length and width is 350 feet. What are its dimensions?

14. The altitude of a certain right triangle is 5 feet more than its base. The hypotenuse is 25 feet. What are its base and altitude?

15. The sum of the base and altitude of a certain right triangle is 17 feet. Its hypotenuse is 13 feet. What are its dimensions?

16. A lawn is 60 by 80 feet. How wide a strip must be cut around it when mowing the grass to have cut half of it?

Hint.—Referring to the figure, it is clear that if w = the number of feet in the width of the border cut, then the dimensions of the uncut part of the lawn are $(60 - 2w)$ and $(80 - 2w)$.
Hence, $(60 - 2w)(80 - 2w) = \frac{1}{2} \cdot 60 \cdot 80$.
Complete the solution.



17. A boy is plowing a field whose dimensions are 20 rods and 40 rods. How wide a border must be cut around it in order to have completed $\frac{3}{4}$ of his plowing?

18. If a certain number be increased by 3 times its reciprocal the sum is $12\frac{1}{4}$. What is the number?

19. The numerator of a certain fraction is 5 less than its denominator. If 3 be added to both numerator and denominator, the new fraction exceeds the original fraction by $\frac{1}{12}$. What is the fraction?

20. A man traveled 84 miles by automobile at a certain rate. By increasing his average rate 7 miles per hour, he made the return trip in 1 hour less time than the time going. What was the rate going and returning?

21. Suppose a motor boat, traveling at the rate of 9 miles per hour in still water, goes 24 miles downstream, and back again in a total of 6 hours. What is the rate of the current of the stream?

22. An auto trip to a town 100 miles away required $7\frac{1}{3}$ hours for the round trip. Going, the party traveled 5 miles more per hour than they did returning. What was their rate going and returning?

23. The denominator of a certain fraction exceeds its numerator by 7. If the numerator be doubled, and the denominator be increased by 2, the sum of the resulting fraction and the original fraction is $\frac{4}{5}$. What is the fraction?

24. An automobile made a trip of 84 miles, of which 9 miles were inside village limits. The average rate outside villages was 12 miles per hour more than inside. If the trip required 3 hours, what was the rate inside and outside villages?

25. A is now five times as old as B. A's age 5 years from now, divided by B's age then, forms a fraction which is $\frac{3}{2}$ of the fraction found by dividing A's age 15 years from now by B's age at the same time. What are their ages now?

EXERCISE 135

Miscellaneous Review

NOTE.—The following examples were proposed as part of a recent examination given to pupils in Regents High Schools of New York.

1. Divide $a^4 + 2a^3 - 6a^2 + 26a - 15$ by $a^2 + 4a - 3$. Check, letting $a = 2$.

2. Factor: *a.* $a^2 - 2a - 48$ *c.* $36m^2 + 60mn + 25n^2$
b. $100x^4 - 49y^6$ *d.* $12c^2 + 7c - 12$

3. Simplify: $\left(4 - \frac{6}{a+1}\right) \div \left(8 - \frac{4a-8}{a^2-1}\right).$

4. Solve and check the system: $\begin{cases} 6x - y = .9 \\ x + 2y = .8 \end{cases}$

5. Simplify: $2\sqrt{108} - 6x\sqrt{5\frac{1}{3}} + \sqrt{3(8x-1)^2}.$

6. Solve and check: $\frac{3x}{2} = x + \frac{x+7}{x-3}.$

7. *a.* Solve for r: $C = \frac{E}{R+r}$

b. Find r to *the nearest tenth* when $C = 15.6$, $R = 4.3$, and $E = 110.7$.

8. Find to the nearest tenth the roots of $x^2 + 2x = \frac{3}{4}$.

9. In a classroom there are 36 desks, some single and some double; if the seating capacity of the room is 42, how many desks of each kind are there?

10. In a certain high school, there were in the:

First year,	690 pupils,	Third year,	420 pupils,
Second year,	750 pupils,	Fourth year,	300 pupils.

Represent this information by a *bar graph.* See page 208.

11. The numerator and denominator of a certain fraction are in the ratio 2 : 3. If 3 be subtracted from the numerator, and 6 from the denominator, the value of the fraction becomes $\frac{4}{3}$. What is the fraction? (*Hint.* — Let $2x$ be the numerator and $3x$ be the denominator.)

12. *a.* The length of a room is L yards, the width is W yards and the height is H yards. (1) Represent the total surface of the four side walls; (2) of the ceiling.

b. From a board b *feet* in length, a piece c *feet* long was cut off from one end, and a piece d *inches* long from the other. How many *inches* in length was the piece which remained?

SUPPLEMENTARY TOPICS

184. Examples and problems involving literal quadratics.

Do these examples as you did the ones in Exercise 133, page 273, using the formula method when you cannot use the factoring method of solution.

EXERCISE 136

Solve for x:

1. $x^2 - 4cx - 5c^2 = 0$

2. $3x^2 - rx - 2r^2 = 0$

3. $x^2 + 2xp + c^2 = 0$

4. $3x^2 + 2mx - n = 0$

5. $s = ax + \frac{1}{2}gx^2$
(From physics)

6. $x^2 - Px + (P - 1) = 0$

7. $x^2 + (a + 1)x + a = 0$

8. $x^2 - (a - b)x - ab = 0$

9. $cx^2 - (c + d)x + d = 0$

10. $\pi x^2 + \pi xl - s = 0$
(From geometry)

11. *a.* The square of a certain number equals the sum of that number and n. What is the number?

b. Using your results as a formula, find the number, when $n = 12$.

12. *a.* If the square of a certain number be increased by n times the number, the result is m. What is the number?

b. Using your results as a formula, what is the number when $n = 8$ and $m = 9$?

13. *a.* There are two consecutive even integers whose product is p. What are they?

b. Using your results as formulæ, find the integers when p is 48.

c. Suppose $p = 9$. Do the results have real meaning according to the statement of the problem?

14. *a.* The length of a certain rectangle exceeds its width by r feet. Its area is s square feet. What are its dimensions?

b. Using your results as formulæ, find the dimensions when $r = 12$ and $s = 13$.

c. Also when $r = 10$ and $s = 100$.

15. *a.* The sum of a certain number and its reciprocal is M. What is the number?

b. Using your results as formulæ, find the number when M is $\frac{13}{6}$.

16. *a.* The sum of the base and altitude of a certain rectangle is p feet, and the area of it is A square feet. What are its dimensions?

b. Using your results as formulæ, what are the dimensions when $p = 40$ and $A = 375$?

185. Quadratic equations having two unknowns.

Example 1. — Find two integers (whole numbers) such that the sum of their squares is 90, and that the larger exceeds twice the smaller by 3.

Solution. — 1. Let $l =$ the larger integer
and $s =$ the smaller.

2. $\therefore\ l^2 + s^2 = 90$ (1)

3. and $l = 2s + 3.$ (2)

4. Eliminate l by the *substitution method of elimination.* (See page 229.)

From (2) $l = 2s + 3.$

Substituting in (1), $(2s + 3)^2 + s^2 = 90.$

5. $\therefore\ 4s^2 + 12s + 9 + s^2 = 90,$ or $5s^2 + 12s - 81 = 0.$

6. Factoring, $(5s + 27)(s - 3) = 0.$

7. $\therefore\ s_1 = 3,$ and $s_2 = -\frac{27}{5}.$

8. When $s = 3, l = 2 \cdot 3 + 3 = 9.$

9. $-\frac{27}{5}$ is unsatisfactory as a solution of the problem, since *integers* were desired.

Check. — When $s = 3$ and $l = 9$, $s^2 + l^2 = 9 + 81 = 90.$
Also $9 = 2 \cdot 3 + 3.$

When there are two unknown numbers, two independent equations must be given. (See page 226.)

In the following list, one of these equations is a quadratic equation and the other is a linear equation. (See page 225.)

Example 2. — Solve the system $\begin{cases} x + y = 2 & (1) \\ xy = -15 & (2) \end{cases}$

Solution. — 1. Solve (1) for y in terms of x. $y = 2 - x$.
2. Substitute in (2), $x(2 - x) = -15$.
3. $\therefore 2x - x^2 = -15$.
4. $\therefore x^2 - 2x - 15 = 0$, or $(x + 3)(x - 5) = 0$.
5. $\therefore x_1 = -3$, and $x_2 = 5$.
6. Since $y = 2 - x$, when $x = -3$, $y = 2 - (-3) = 5$;
when $x = 5$, $y = 2 - 5 = -3$.

Check. — Each *solution* (see page 225) should be checked by substituting it in both equations, (1) and (2).

EXERCISE 137

1. $\begin{cases} x - y = 3 \\ xy = 10 \end{cases}$

2. $\begin{cases} x + y = -2 \\ xy = -24 \end{cases}$

3. $\begin{cases} 2x + y = 5 \\ xy = 3 \end{cases}$

4. $\begin{cases} a^2 + b^2 = 13 \\ a - 2b = -4 \end{cases}$

5. $\begin{cases} x^2 - y^2 = -7 \\ 2x - y = -10 \end{cases}$

6. $\begin{cases} cd = -10 \\ 2c + 3d = 11 \end{cases}$

7. $\begin{cases} 3s + t = 0 \\ 9s^2 + 4t^2 = 5 \end{cases}$

8. $\begin{cases} 4m^2 + n^2 = 10 \\ 2m + n = 4 \end{cases}$

9. $\begin{cases} x^2 - 2xy - y^2 = 17 \\ x - 2y = 7 \end{cases}$

10. $\begin{cases} r^2 + rs + s^2 = 5\frac{1}{4} \\ r - 2s = 1 \end{cases}$

11. Find two numbers whose sum is 10 and whose product is 21.

12. Find two numbers whose sum is 11 and the sum of whose squares is 61.

13. One number exceeds another number by 3. The square of the larger exceeds twice the square of the smaller by 14. What are the numbers?

14. One number exceeds 3 times a smaller number by 1. The larger, increased by the square of the smaller, is 11. What are the numbers?

15. The perimeter of a certain rectangle is 30 inches. Its area is 36 square inches. What are its dimensions?

16. The units' digit of a certain number exceeds its tens' digit by 5. If the number itself be multiplied by its tens' digit, the product is 114. What is the number?

17. The sum of the squares of the two digits of a number is 37. If the digits be reversed, 4 times the new number exceeds the original number by 3. What is the number?

18. The perimeter of a certain parallelogram is 36 inches. The square of the longer side exceeds 6 times the square of the shorter side by 19 inches. How long are its sides?

19. One of two angles exceeds the other by 10°. If each of the angles be multiplied by its complement (see page 23), the two products are equal. What are the angles?

20. Two angles are supplementary. The square of the larger exceeds 100 times the smaller by 2000. What are the angles?

21. The rate of a certain passenger train exceeds that of a freight train by 15 miles per hour. The passenger train requires 3 hours less than the freight train for a trip of 140 miles. What are their rates?

22. A and B together can do a certain piece of work in 8 days. It takes B 12 days longer than A to do the work alone. How long does it take each to do it alone?

23. The rate of a certain passenger train exceeds that of a freight train by 18 miles per hour. The passenger train requires 1 hour less time for a trip of 160 miles than the freight train requires for a trip of 110 miles. What is the rate of each?

24. It takes A 2 days longer to do a certain piece of work than it does B. They can do the work if A works alone 5 days and B works 4 days. How long does it take each to do it alone?

GRAPHICAL SOLUTION OF EQUATIONS WITH ONE VARIABLE

186. Definition of function. Consider $3x - 12$. For every value of x, there is a definite value of $3x - 12$.

$3x - 12$ is said to be a function of x.

One number (or number expression) is a function of another number when there is a definite value of the first for every value of the second.

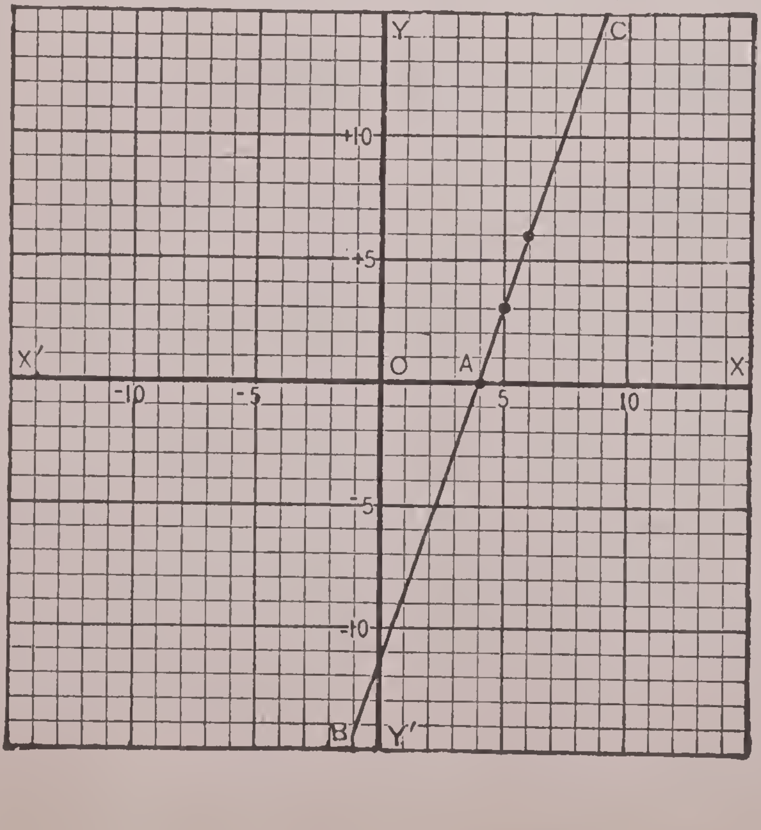
Thus, $5 \cdot b$ represents the area of a rectangle of base b and altitude 5. For every value of b there is a definite value of the area of $5b$. Consequently, the area of a rectangle of altitude 5 is a function of the base.

Every expression involving x is a function of x.

Thus, $2x$ is a function of x, — a *first degree function.*
x^2 is a function of x, — a *second degree function.*
$x^2 - 3x + 2$ is a function of x, — a *second degree function.*

187. Graph of a function. Consider again the function $3x - 12$. As a matter of convenience, let y represent the value of $3x - 12$.

When $x =$	0	2	3	4	5	6	etc.
Then y (or $3x - 12$) $=$	-12	-6	-3	0	$+3$	$+6$	etc.



Use each pair of numbers as the coördinates of a point; thus, the point $(2, -6)$. The *abscissa*, 2, is the value of x; and the ordinate, -6, is the corresponding value of the function.

Connect the points by a smooth line. The graph is at the left.

188. Equations having one unknown can be solved graphically.

Example 1. — Solve the equation $3x = 12$ graphically.

Solution. — 1. $3x = 12$.

2. S_{12} $3x - 12 = 0$.

3. Draw the graph of the function $3x - 12$.

(This was done in § 187, page 282. Look at that graph.)

4. Observe point A. For it, the value of the function (that is, the value of y) is 0. The value of x which gives that value of y is 4. That is, when $x = 4$, $3x - 12$ is 0.

5. 4 is the required root of the equation.

NOTE 1. — Remember that we seek a value of x which makes $3x - 12$ equal zero when we solve the equation $3x - 12 = 0$. That value of x in the graphical solution is the abscissa of the point where the graph crosses the x-axis.

NOTE 2. — This graphical method of solving an equation having only one unknown is studied further in more advanced courses in algebra.

Rule. — To solve an equation containing one unknown graphically:

1. Clear the equation of fractions, collect like terms, and transpose all terms to the left side of the equation.

2. Represent by y the function obtained in Step 1.

3. Select values of the variable given in the equation and find the corresponding values of y.

4. Use each pair of values obtained in Step 3 as the coordinates of a point; locate the points; draw the graph, making the vertical axis the y-axis, or the function-axis.

5. Each point at which the graph crosses the horizontal axis represents a zero value of y, and the abscissa of such a point is the value of the variable which gives the zero value of y. Each such abscissa is a root of the given equation.

Example 2. — Solve the equation $\frac{x}{3} - \frac{1}{3} = \frac{2}{x}$ graphically.

Solution. — 1. $\frac{x}{3} - \frac{1}{3} = \frac{2}{x}.$

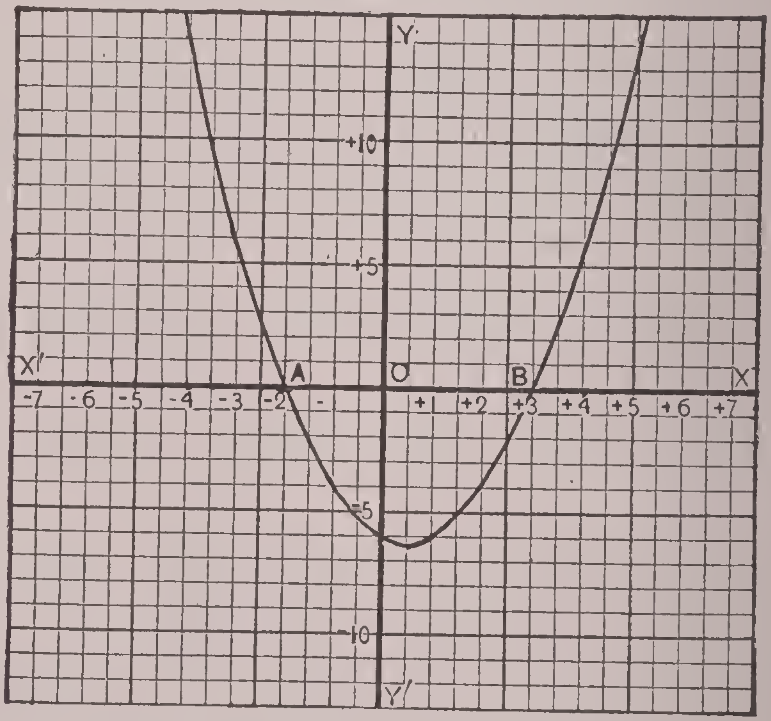
2. M_{3x} $x^2 - x = 6$, or $x^2 - x - 6 = 0$.

3. Let $y =$ the function $x^2 - x - 6$.

4.

When $x =$	0	+ 1	+ 2	+ 4	+ 5	− 1	− 2	− 3	− 4
Then $y =$	− 6	− 6	− 4	+ 6	+ 14	− 4	0	+ 6	+ 14

5. The graph follows.



6. The graph crosses the horizontal axis at A and B.

At A, $x = -2$. At B, $x = +3$. These are the roots of the given equation.

Check. — When $x = -2$ does $\frac{-2}{3} - \frac{1}{3} = \frac{2}{-2}$?

Does $-\frac{2}{3} - \frac{1}{3} = -1$? Yes.

Similarly, + 3 checks.

Note. — Do not say "draw the graph of the equation." Say "solve the equation graphically." To do so, *you draw the graph of a certain function derived from the equation.*

EXERCISE 138

Solve the following equations graphically:

1. $3x = 2$

2. $x^2 = 25$

3. $x^2 + x = 12$

4. $x^2 = 10 - 3x$

5. $x^2 - x - 12 = 0$

6. $x^2 - 4x + 4 = 0$

189. Imaginary roots of a quadratic equation.

Example. — Solve the equation $x^2 - 2x + 5 = 0$.

Solution. — 1. Use the formula method of solution.

$$a = 1;\ b = -2;\ c = 5.$$

2. $$\therefore x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1}$$

3. $$= \frac{2 \pm \sqrt{4 - 20}}{2} = \frac{2 \pm \sqrt{-16}}{2}.$$

What does $\sqrt{-16}$ mean?

-4 is not the square root of -16, since $(-4)^2 = +16$.
$+4$ is not the square root of -16, since $(+4)^2 = +16$.

No number, studied so far, will produce a negative result when it is squared. **$\sqrt{-16}$ is a new kind of number, — an imaginary number.**

An **imaginary number** is the indicated square root of a negative number, as $\sqrt{-6}$; $\sqrt{-3}$; $\sqrt{-\frac{1}{2}}$.

The numbers studied previously, in contrast to the imaginary numbers, are called **real numbers.**

190. Every imaginary number can be expressed as the product of a real number and $\sqrt{-1}$.

$\sqrt{-1}$ is indicated by i, and is called the **imaginary unit.**

Thus,
$$\sqrt{-16} = \sqrt{16(-1)} = \pm 4\sqrt{-1} = \pm 4i.$$
$$\sqrt{-a^2} = \sqrt{a^2(-1)} = \pm a\sqrt{-1} = \pm ai.$$
$$\sqrt{-5} = \sqrt{5(-1)} = \pm\sqrt{-1 \cdot 5} = \pm i\sqrt{5}.$$

NOTE. — The symbol i was introduced by Euler, one of the greatest mathematicians of the eighteenth century.

EXERCISE 139

Express the following in terms of i:

1. $\sqrt{-4}$
2. $\sqrt{-9}$
3. $\sqrt{-169}$
4. $\sqrt{-81}$
5. $\sqrt{-64}$
6. $\sqrt{-25x^2}$
7. $\sqrt{-36m^4}$
8. $\sqrt{-121n^2}$
9. $\sqrt{-400x^2}$
10. $\sqrt{-196x^4}$
11. $\sqrt{-\frac{1}{9}}$
12. $\sqrt{-\frac{1}{16}}$
13. $\sqrt{-\frac{4}{25}a^2}$
14. $\sqrt{-7}$
15. $\sqrt{-12}$
16. $\sqrt{-27}$
17. $\sqrt{-48}$
18. $\sqrt{-32m^2}$
19. $\sqrt{-\frac{9}{16}}$
20. $\sqrt{-75}$

21. Simplify $\sqrt{-\frac{27}{4}}$.

Solution. — $\sqrt{-\frac{27}{4}} = \sqrt{\frac{9\cdot 3\cdot(-1)}{4}} = \pm\frac{3}{2}\cdot\sqrt{3\cdot -1} = \pm\frac{3}{2}i\sqrt{3}$.

NOTE. — $\sqrt{3\cdot -1}$ really gives $\sqrt{3}\cdot i$, but the factor i is always written in front of any radical which multiplies it.

22. $\sqrt{-\frac{5}{9}}$
23. $\sqrt{-\frac{3}{16}}$
24. $\sqrt{-\frac{7}{4}}$
25. $\sqrt{-\frac{16}{9}}$
26. $\sqrt{-\frac{32}{9}}$
27. $\sqrt{-\frac{20}{9}}$
28. $\sqrt{-\frac{27}{25}}$
29. $\sqrt{-\frac{27}{16}}$
30. $\sqrt{-\frac{125}{81}}$

31. Simplify $-\frac{5}{2}\pm\sqrt{-\frac{27}{4}}$.

Solution. — 1. $$-\frac{5}{2}\pm\sqrt{-\frac{27}{4}} = -\frac{5}{2}\pm\sqrt{\frac{3\cdot 9\cdot(-1)}{4}}$$

2. $$= -\frac{5}{2}\pm\frac{3i\sqrt{3}}{2}$$

3. $$= \frac{-5\pm 3i\sqrt{3}}{2}.$$

NOTE. — The numbers in Examples 1–30 are called *pure imaginaries*. The numerator of the result in Example 31 is called a *complex number*.

32. $\frac{1}{3}\pm\sqrt{-\frac{5}{9}}$
33. $\frac{3}{2}\pm\sqrt{-\frac{3}{4}}$
34. $-\frac{2}{5}\pm\sqrt{-\frac{6}{25}}$
35. $\frac{5}{4}\pm\sqrt{-\frac{9}{16}}$
36. $-\frac{3}{8}\pm\sqrt{-\frac{27}{64}}$
37. $\frac{2}{7}\pm\sqrt{-\frac{12}{49}}$
38. $-\frac{5}{6}\pm\sqrt{-\frac{7}{36}}$
39. $\frac{9}{10}\pm\sqrt{-\frac{27}{100}}$
40. $\frac{5}{9}\pm\sqrt{-\frac{8}{81}}$

191. Solving quadratics which have imaginary roots.

Example. — Solve the equation $x^2 + x + 2 = 0$.

Solution. — 1. $a = 1;\ b = 1;\ c = 2.$

2. $$\therefore x = \frac{-1 \pm \sqrt{1-8}}{2} = \frac{-1 \pm \sqrt{-7}}{2} = \frac{-1 \pm i\sqrt{7}}{2}.$$

3. $$\therefore x_1 = \frac{-1 + i\sqrt{7}}{2}; \qquad x_2 = \frac{-1 - i\sqrt{7}}{2}.$$

We shall also solve this equation graphically.

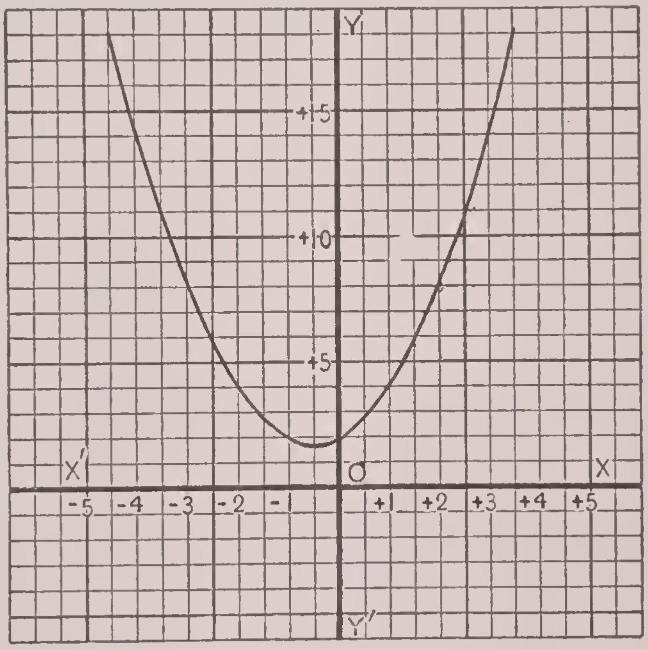
Solution. — 1. Let $y = x^2 + x + 2$.

2.

When $x =$	0	+ 1	+ 2	+ 3	− 1	− 2	− 3	− 4
then $y =$	+ 2	+ 4	+ 8	+ 14	+ 2	+ 4	+ 8	+ 14

3. The graph has the same shape as the graphs obtained when solving other quadratic equations; *but the graph does not cross* the horizontal axis at all. Hence, y or $x^2 + x + 2$ is never zero for any real value of x.

This is characteristic of the graph of a quadratic which has *imaginary* roots.



EXERCISE 140

Solve the following equations either by completing the square or by the formula. Draw the graphs of the first three equations.

1. $x^2 + x + 3 = 0$

2. $x^2 + 2x + 2 = 0$

3. $x^2 - 3x + 4 = 0$

4. $2x^2 - x + 1 = 0$

5. $3y^2 - 2y + 2 = 0$

6. $2a^2 - 3a + 2 = 0$

7. $3t^2 - 4t + 2 = 0$

8. $4r^2 - 5r + 2 = 0$

9. $7s^2 + 6s + 3 = 0$

10. $6c^2 - 5c + 2 = 0$

XV. SPECIAL PRODUCTS AND FACTORING

ADVANCED TOPICS

192. Product of two polynomials reducible to the type forms $(a + b)(a - b)$ and $(ax + b)(cx + d)$ studied in Chapter VIII.

You have learned that $(a + b)(a - b) = a^2 - b^2$, and that this relation is true for all values of a and b.

Thus, if $a = 5x$ and $b = 2y$,

$$(5x + 2y)(5x - 2y) = (5x)^2 - (2y)^2 = 25x^2 - 4y^4.$$

If $a = 17$ and $b = 13$, $(17 + 13)(17 - 13) = 17^2 - 13^2$

$$= 289 - 169 = 120.$$

If now, $a = (x + y)$ and $b = (z + w)$, then again,

$$\{(x + y) + (z + w)\}\{(x + y) - (z + w)\} = (x + y)^2 - (z + w)^2$$
$$= (x^2 + 2xy + y^2) - (z^2 + 2zw + w^2)$$
$$= x^2 + 2xy + y^2 - z^2 - 2zw - w^2.$$

In this example, $(x + y)$ takes the place of a, and $(z + w)$ takes the place of b.

By properly grouping terms, certain polynomials can be changed to type forms whose product can be found mentally.

Example. — Find $(r - s + t - u)(r + s + t + u)$.

Solution. — Trial *a*. 1. Grouping r and s, and t and u,

$$(r - s + t - u)(r + s + t + u)$$
$$= \{(r - s) + (t - u)\}\{(r + s) + (t + u)\}.$$

$(r - s)$ appears in the first set of braces, and $(r + s)$ in the second. *These are not the same number expression. Therefore this is an unsatisfactory grouping.*

Trial *b*. 1. Group r and t, and s and u.

$$(r - s + t - u)(r + s + t + u)$$
$$= \{(r + t) - (s + u)\}\{(r + t) + (s + u)\}.$$

Now, $(r + t)$ appears in both sets of braces, and also $(s + u)$.

2. This is the type form $(a - b)(a + b)$ where $a = (r + t)$ and $b = (s + u)$.

3. $$\therefore \{(r + t) - (s + u)\}\{(r + t) + (s + u)\}$$

4. $= (r + t)^2 - (s + u)^2$
5. $= (r^2 + 2\,rt + t^2) - (s^2 + 2\,su + u^2)$
6. $= r^2 + 2\,rt + t^2 - s^2 - 2\,su - u^2.$

Check. — Let $r = 1$, $s = 2$, $t = 1$, $u = 2$.

Then

$$\left.\begin{array}{l}(r - s + t - u) = 1 - 2 + 1 - 2 = -2\\(r + s + t + u) = 1 + 2 + 1 + 2 = 6\end{array}\right\}\text{ and } (-2)\cdot(6) = -12.$$

Also, $r^2 + 2\,rt + t^2 - s^2 - 2\,su - u^2 = 1 + 2 + 1 - 4 - 8 - 4 = -12.$

EXERCISE 141

Find by use of the special products:

1. $\{(a - b) + 6\}\{(a - b) - 6\}$
2. $\{(c + d) - 3\,e\}\{(c + d) + 3\,e\}$
3. $\{8 - (a + b)\}\{8 + (a + b)\}$
4. $\{2\,r - (s - t)\}\{2\,r + (s - t)\}$
5. $\{(a - 3\,d) + 4\,c\}\{(a - 3\,d) - 4\,c\}$
6. $(x + y + z)(x + y - z)$
7. $(r - s - t)(r - s + t)$
8. $(m - p + n)(m + p + n)$
9. $(2\,a - b + c)(2\,a + b - c)$
10. $(x^2 + x - 1)(x^2 - x + 1)$
11. $(a^2 - ab + b^2)(a^2 + ab + b^2)$
12. $(r - 2\,s + 3\,t)(r + 2\,s + 3\,t)$
13. $(a + b + c - d)(a + b - c + d)$
14. $(x + y - z - w)(x - y + z - w)$
15. $(m - r + n + p)(m + r + n - p)$
16. Find $(x^2 + 1 + 2\,x)(x^2 - 5 + 2\,x)$

Solution. — 1. $(x^2 + 1 + 2\,x)(x^2 - 5 + 2\,x)$

2. $= \{(x^2 + 2\,x) + 1\}\{(x^2 + 2\,x) - 5\}.$

3. $(x^2 + 2\,x)$ is in both braces; but 1 and 5 are different numbers. This is a product like $(y + 1)(y - 5)$, which equals $y^2 - 4\,y - 5$ where $(x^2 + 2\,x)$ takes the place of y.

(Continued on page 290.)

4. $\therefore \{(x^2 + 2\,x) + 1\}\{(x^2 + 2\,x) - 5\} = (x^2 + 2\,x)^2 - 4(x^2 + 2\,x) - 5$

5. $= (x^4 + 4\,x^3 + 4\,x^2) - 4(x^2 + 2\,x) - 5$

6. $= x^4 + 4\,x^3 + 4\,x^2 - 4\,x^2 - 8\,x - 5$

7. $= x^4 + 4\,x^3 - 8\,x - 5.$

Check. — Let $x = 1$.

$$\left.\begin{aligned} x^2 + 1 + 2\,x &= 1 + 1 + 2 = 4 \\ x^2 - 5 + 2\,x &= 1 - 5 + 2 = -2 \end{aligned}\right\}\text{ and } 4(-2) = -8.$$

Also, $x^4 + 4\,x^3 - 8\,x - 5 = 1 + 4 - 8 - 5 = -8.$

Find:

17. $\{(a + b) + 2\}\{(a + b) + 3\}$

18. $\{(x - y) - 3\}\{(x - y) + 7\}$

19. $\{6 - (m + n)\}\{8 + (m + n)\}$

20. $\{a + (b + c)\}\{a + (b + c)\}$

21. $\{(x - y) - z\}\{(x - y) - z\}$

22. $(a + b - 4)(a + b + 6)$

23. $(x + y + 5)(x + y - 8)$

24. $(7 + p + q)(8 - p - q)$

25. $(r - s - t)(r - s - t)$

26. $(x + y - z)(x + y - z)$

27. $\{2\,x - (r + s)\}\{3\,x + (r + s)\}$

28. $\{2\,x - y + z\}\{2\,x - y - 5\,z\}$

29. $\{3\,m - 2(a + b)\}\{2\,m + 3(a + b)\}$

30. $\{4\,p - 5\,r - 5\,s\}\{3\,p + 2\,r + 2\,s\}$

193. Factoring generalizations of the type forms studied in Chapter VIII.

Example 1. — Just as $x^2 - y^2 = (x + y)(x - y)$

$$\begin{aligned} \text{so } (m - n)^2 - 25\,a^2 &= \{(m - n) - 5\,a\}\{(m - n) + 5\,a\} \\ &= \{m - n - 5\,a\}\{m - n + 5\,a\}. \end{aligned}$$

Example 2. — Just as $x^2 - 3\,x - 88 = (x - 11)(x + 8)$

$$\begin{aligned} \text{so } (a - 2\,b)^2 - 3(a - 2\,b) - 88 &= \{(a - 2\,b) - 11\}\{(a - 2\,b) + 8\} \\ &= \{(a - 2\,b - 11\}\{a - 2\,b + 8\}. \end{aligned}$$

EXERCISE 142

Factor the following expressions:

1. $(a - b)^2 - c^2$

2. $x^2 - (z - y)^2$

3. $(2\,a - b)^2 - 9\,c^2$

4. $16\,s^2 - (t + w)^2$

5. $(m - n)^2 - 8(m - n) + 15$
6. $(r + s)^2 + 4(r + s) - 77$
7. $(2t - w)^2 - 3s(2t - w) - 10s^2$
8. $4x^2 - 4x(y + z) + (y + z)^2$
9. $6x^2 + x(y + z) - 2(y + z)^2$
10. $4(m + n)^2 - 5(m + n)p - 6p^2$
11. $4(a + b)^2 - 12(a + b) + 9$
12. $16(c - d)^2 - 9(x - y)^2$
13. $(x^2 + y^2)^2 - 4x^2y^2$
14. $(a^2 - 2a)^2 + 4(a^2 - 2a) + 3$
15. $(9a^2 + 1)^2 - 36a^2$
16. $(x^2 + 2x)^2 + 2(x^2 + 2x) + 1$
17. $(r^2 - 3)^2 - (r + 1)^2$
18. $(x^2 + 1)^2 - 4x^2$
19. $(x^2 + 2x)^2 - 2(x^2 + 2x) - 3$
20. $(a^2 - 3a)^2 - 2(a^2 - 3a) - 8$

194. Polynomials reducible to the difference of two squares.

Example. — Factor $a^2 - c^2 + b^2 - d^2 - 2cd - 2ab$.

Solution. — 1. It is natural to group $-2ab$ with a^2, and that at once suggests grouping b^2 with them.

$\therefore\ a^2 - c^2 + b^2 - d^2 - 2cd - 2ab$
2. $= (a^2 - 2ab + b^2) - (c^2 + 2cd + d^2)$
3. $= (a - b)^2 - (c + d)^2$
4. $= \{(a - b) + (c + d)\}\{(a - b) - (c + d)\}$
5. $= \{a - b + c + d\}\{a - b - c - d\}$.

Check. — Let $a = 1$, $b = 2$, $c = 1$, $d = 2$.

$a^2 - c^2 + b^2 - d^2 - 2cd - 2ab = 1 - 1 + 4 - 4 - 4 - 4 = -8.$
$\{a - b + c + d\{\{a - b - c - d\} = \{1 - 2 + 1 + 2\}\{1 - 2 - 1 - 2\}$
$= 2(-4)$, or -8.

Note. — A polynomial of four terms can be factored in this way if three terms make a perfect square trinomial and if the fourth term is a perfect square monomial, preceded by a minus sign.

EXERCISE 143

Factor:

1. $a^2 + 2ab + b^2 - c^2$
2. $m^2 - 2mn + n^2 - 4p^2$
3. $x^2 + 6x + 9 - y^2$
4. $t^2 - r^2 - 2rs - s^2$
5. $4c^2 - a^2 + 2ab - b^2$
6. $9x^2 - 6xy - z^2 + y^2$
7. $16a^2 + b^2 + 8ab - c^2$
8. $9r^2 - 4s - 4 - s^2$
9. $25t^2 - 4x^2 + 10t + 1$
10. $8x + 49y^2 - x^2 - 16$
11. $6t + 16m^2 - 9 - t^2$
12. $36 - 12ab - a^2 - 36b^2$
13. $81z^2 + 20w - 4w^2 - 25$
14. $4xy - z^2 + x^2 + 4y^2$
15. $x^2 + y^2 - 2xy + 2cd - c^2 - d^2$
16. $6ab + a^2 - x^2 + 9b^2 - 4x - 4$
17. $4a^2 + 1 - b^2 - 4a - 9 - 6b$
18. $r^2 - 9s^2 + 25 + 6st - 10r - t^2$
19. $4a^2 - x^2 + 2xy - 12ab - y^2 + 9b^2$
20. $2cd + 25a^2 - c^2 - 10ab - d^2 + b^2$

195. Trinomials reducible to the difference of two squares.

Type form: $x^4 + x^2y^2 + y^4$

Example. — Factor $64a^4 - 64a^2m^2 + 25m^4$.

Solution. — 1. *A perfect square* containing $64a^4$ and $25m^4$ is $64a^4 - 80a^2m^2 + 25m^4$. It can be obtained from $64a^4 - 64a^2m^2 + 25m^4$ by subtracting $16a^2m^2$. Then we must also add $16a^2m^2$ to avoid changing the value of $64a^4 - 64a^2m^2 + 25m^4$. That is

2. $64a^4 - 64a^2m^2 + 25m^4 = (64a^4 - 80a^2m^2 + 25m^4) + 16a^2m^2$
$$= (8a^2 - 5m^2)^2 + (4am)^2$$

But this is not factorable, since it is the sum of two squares.

3. *Another perfect square* containing $64a^4$ and $25m^4$ is $64a^4 + 80a^2m^2 + 25m^4$. It can be obtained from $64a^4 - 64a^2m^2 + 25m^4$ by adding to it $144a^2m^2$. Then we must also subtract $144a^2m^2$ to avoid changing the value of the given expression. That is

$$64a^4 - 64a^2m^2 + 25m^4$$
$$= (64a^4 + 80a^2m^2 + 25m^4) - 144a^2m^2$$

4. $= (8a^2 + 5m^2)^2 - (12am)^2$

5. $= \{(8a^2 + 5m^2) + 12am\}\{(8a^2 + 5m^2) - 12am\}$

6. $= \{8a^2 + 5m^2 + 12am\}\{8a^2 + 5m^2 - 12am\}$

Check. — Let $a = 1$ and $b = 1$; and $m = 1$.

$$64\,a^4 - 64\,a^2m^2 + 25\,m^4 = 64 - 64 + 25, \text{ or } 25$$

$$\{8\,a^2 + 5\,m^2 + 12\,am\}\{8\,a^2 + 5\,m^2 - 12\,am\}$$

$$= \{8 + 5 + 12\}\{8 + 5 - 12\}$$

$$= 25 \cdot (1), \text{ or } 25.$$

NOTE. — If you are far sighted or fortunate you may try Steps 3 and 4, first, and then Steps 1 and 2 are, of course, unnecessary, and do not appear in your solution.

EXERCISE 144

Factor:

1. $a^4 + a^2b^2 + b^4$

2. $m^4 + m^2 + 1$

3. $x^4 + 4\,x^2 + 16$

4. $y^4 + 9\,y^2 + 81$

5. $z^4 + 6\,z^2w^2 + 25\,w^4$

6. $r^4 + 3\,r^2s^2 + 36\,s^4$

7. $9\,m^4 + 2\,m^2t^2 + t^4$

8. $16\,a^4 - 12\,a^2 + 1$

9. $9\,x^4 - 10\,x^2 + 1$

10. $9\,a^4 + 2\,a^2b^2 + b^4$

11. $36\,t^4 - 16\,t^2 + 1$

12. $25\,x^4 - 19\,x^2y^2 + y^4$

13. $4\,a^4 + 8\,a^2 + 9$

14. $9\,x^4 + 20\,x^2 + 16$

15. $4\,a^4 + 11\,a^2b^2 + 25\,b^4$

16. $9\,c^4 - 34\,c^2d^2 + 25\,d^4$

17. $16\,a^4 - 28\,a^2b^2 + 9\,b^4$

18. $9\,r^4 - 43\,r^2t^2 + 49\,t^4$

19. $16\,x^4 + 4\,x^2y^2 + 25\,y^4$

20. $4\,m^4 + 19\,m^2n^2 + 49\,n^4$

Factor similarly the following binomials:

21. $x^4 + 4$

22. $4\,a^4 + 1$

23. $c^4 + 4\,d^4$

24. $4\,x^4 + y^4$

25. $r^4 + 64\,t^4$

26. $x^4 + 324$

196. Polynomials factored by grouping.

Example 1. — Just as $ax + bx = x(a + b)$
so $a(m + n) + b(m + n) = (m + n)(a + b)$.

In this example $(m + n)$ takes the place of x.

Example 2. — Factor $6\,x^3 - 15\,x^2 - 8\,x + 20$.

Solution. — 1. $6\,x^3 - 15\,x^2 - 8\,x + 20$

2. $= 3\,x^2(2\,x - 5) - 4(2\,x - 5)$

3. $= (3\,x^2 - 4)(2\,x - 5)$.

Check. — Let $x = 1$.

$6x^3 - 15x^2 - 8x + 20 = 6 - 15 - 8 + 20$, or 3.

$(3x^2 - 4)(2x - 5) = (3 - 4)(2 - 5) = (-1)(-3)$, or $+3$.

EXERCISE 145

Factor:

1. $a(x + y) - b(x + y)$
2. $2m(r - s) + 3n(r - s)$
3. $p(2a - b) - q(2a - b)$
4. $2x(y - z) - 3w(y - z)$
5. $5(m + x) - r(m + x)$
6. $ac + ad + bc + bd$
7. $rt + rw - st - sw$
8. $mx - my - nx + ny$
9. $x^3 - 2x^2 + x - 2$
10. $2a^3 - 3a^2 - 3 + 2a$
11. $2a - b - 2ac + bc$
12. $3 - 6b^2 + 5b - 10b^3$
13. $8t^3 + 2t - 4t^2 - 1$
14. $15am - 6bm + 5ax - 2bx$
15. $a^3 - a^2b + ab^2 - b^3$
16. $c^3 + c^2d - d^2 - cd$
17. $24a^3 - 20a^2 - 5 + 6a$
18. $3mx + 6nx + my + 2ny$
19. $2x^3 - 6xy^2 + x^2y - 3y^3$
20. $40x^3 - 42 - 35x^2 + 48x$
21. $ax - ay + az + bx - by + bz$
22. $ax - bx - ay - az + by + bz$
23. $2ac - bc + 4a - 2b + 6 + 3c$
24. $2x^3 - x^2y + 2x^2 - xy + 2x - y$
25. $2am - 3bm - 2an + 3bn - 6bp + 4ap$
26. $2rx + 2rz - sx + 3sy - 6ry - sz$
27. $10ap - 8aq + 15cp + 4bq - 12cq - 5bp$

197. The sum or difference of like powers of two numbers.

Development. — 1. Make a table like the following and fill in the quotients and remainders.

NOTE. — A good class exercise is to place this table on the blackboard. Time can be saved by distributing the division problems among the members of the class.

		DIVIDEND	DIVISOR	QUOTIENT	REMAINDER
a.	Sum of odd powers	$x^3 + y^3$ $x^5 + y^5$ $x^3 + y^3$ $x^5 + y^5$	$x + y$ $x + y$ $x - y$ $x - y$		
b.	Sum of even powers	$x^2 + y^2$ $x^4 + y^4$ $x^2 + y^2$ $x^4 + y^4$	$x + y$ $x + y$ $x - y$ $x - y$		
c.	Difference of odd powers	$x^3 - y^3$ $x^5 - y^5$ $x^3 - y^3$ $x^5 - y^5$	$x - y$ $x - y$ $x + y$ $x + y$		
d.	Difference of even powers	$x^2 - y^2$ $x^4 - y^4$ $x^2 - y^2$ $x^4 - y^4$	$x - y$ $x - y$ $x + y$ $x + y$		

Examination of the results will disclose the following rules.

Rules. — I. When n is a positive integer.

1. When n is odd, $x^n + y^n$ is exactly divisible by $x + y$, but not by $x - y$.

2. When n is even, $x^n + y^n$ is not exactly divisible by either $x + y$ or $x - y$.

3. When n is odd, $x^n - y^n$ is exactly divisible by $x - y$, but not by $x + y$.

4. When n is even, $x^n - y^n$ is exactly divisible by $x - y$, and also by $x + y$.

II. As to the quotients in all cases:

1. When $x - y$ is the divisor, the signs of the quotient are all plus.

2. When $x + y$ is the divisor, the signs of the quotient are alternately plus and minus, the first term being plus.

3. The exponent of x in the first term is 1 less than its exponent in the dividend, and decreases by 1 in each succeeding term until it becomes 1 in the next to the last term. x does not appear in the last term.

4. y does not appear in the first term. The exponent of y in the second term is 1, and increases by 1 in each succeeding term until it becomes 1 less than the exponent of y in the dividend.

III. **Two particularly important formulæ.**

1. $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ **Rules I–3; II–1, 3, 4.**

2. $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ **Rules I–1; II–2, 3, 4.**

Example 1. — Factor $8a^3 - 27b^6$.

Solution. — 1. $8a^3 - 27b^6 = (2a)^3 - (3b^2)^3$.

2. This is the difference of two cubes, and may be solved by the formula III–1.

$\therefore\ 8a^3 - 27b^6 = (2a - 3b^2)[(2a)^2 + (2a)\cdot(3b^2) + (3b^2)^2]$

3. $= (2a - 3b^2)(4a^2 + 6ab^2 + 9b^4)$.

Check. — Let $a = 1$; $b = 1$. $8a^3 - 27b^6 = 8 - 27$, or -19.

$(2a - 3b^2)(4a^2 + 6ab^2 + 9b^4) = (2 - 3)(4 + 6 + 9)$, or $-1 \cdot 19$
$= -19$.

Example 2. — Factor $32x^5 + 243y^{10}$.

Solution. — 1. $32x^5 + 243y^{10} = (2x)^5 + (3y^2)^5$.

2. This is the sum of two odd powers. By I–1, and II–2, 3, 4,

$$32x^5 + 243y^{10}$$
$$= (2x + 3y^2)[(2x)^4 - (2x)^3(3y^2) + (2x)^2(3y^2)^2 - (2x)(3y^2)^3 + (3y^2)^4]$$

3. $= (2x + 3y^2)(16x^4 - 8x^3 \cdot 3y^2 + 4x^2 \cdot 9y^4 - 2x \cdot 27y^6 + 81y^8)$

4. $= (2x + 3y^2)(16x^4 - 24x^3y^2 + 36x^2y^4 - 54xy^6 + 81y^8)$.

Check. — Let $x = 1$; $y = 1$. Then $32x^5 + 243y^{10} = 32 + 243$, or 275.

$$(2x + 3y^2)(16x^4 - 24x^3y^2 + 36x^2y^4 - 54xy^6 + 81y^8)$$
$$= (2 + 3)(16 - 24 + 36 - 54 + 81) = 5(55), \text{ or } 275.$$

The method of factoring binomials of the form $x^n \pm y^n$ given in this section must be used frequently. However, **if the prime factors of a binomial of this form are wanted, proceed as follows:**

Whenever the binomial is the difference of two even powers, treat it as the difference of two squares.

Example 3. — $x^6 - y^6 = (x^3 - y^3)(x^3 + y^3)$
$= (x - y)(x^2 + xy + y^2)(x + y)(x^2 - xy + y^2).$

When the binomial is the sum of two powers, try to express it as the sum of two cubes; if not cubes, the sum of two fifth powers; etc.

Example 4. — 1. $x^9 + y^9 = (x^3)^3 + (y^3)^3$
2. $= (x^3 + y^3)\,[(x^3)^2 - (x^3)(y^3) + (y^3)^2]$
3. $= (x^3 + y^3)(x^6 - x^3y^3 + y^6)$
4. $= (x + y)(x^2 - xy + y^2)(x^6 - x^3y^3 + y^6).$

NOTE 1. — Observe x^9 is thought of as $(x^3)^3$; similarly, think of x^{10} as $(x^5)^2$ or as $(x^2)^5$; x^6 as $(x^3)^2$ or $(x^2)^3$; x^{12} as $(x^4)^3$ or as $(x^3)^4$.

NOTE 2. — In Example 3, prime factors of $x^6 - y^6$ are found.

a. By Rule I–4, $(x - y)$ is a factor of $x^6 - y^6$. Find the other factor by II–1, 3, 4. Is this factor prime?

b. Referring to Example 3, above, find the product of $(x + y)(x^2 - xy + y^2)(x^2 + xy + y^2)$. Compare it with the second factor in part *a*.

c. By Rule I–4, $(x + y)$ is a factor of $x^6 - y^6$. Find the other factor by II–2, 3, 4.

EXERCISE 146

Factor completely the following expressions, if possible.

1. $x^3 - 8y^3$
2. $b^6 - 27c^3$
3. $c^6 + 8d^3$
4. $8r^6 + 1$
5. $8a^3 - 27b^3$
6. $27m^3 + n^6$
7. $x^6 - 125y^3$
8. $64 + m^9$
9. $27a^3 + 64b^3$
10. $8c^6 - 125d^3$
11. $x^4 + 16y^4$
12. $x^4 - y^4$
13. $x^6 + y^6$
14. $x^8 + y^8$
15. $x^8 - y^8$
16. $x^{10} + y^{10}$
17. $x^{10} - y^{10}$
18. $x^{12} - y^{12}$
19. $x^4 - 16$
20. $x^5 - 32$
21. $243 + x^{10}$
22. $32a^5 + 1$
23. $x^5 - y^{10}$
24. $r^3s^3 - 27t^3$
25. $a^7 - b^7$
26. $m^{10} - 243n^5$
27. $\frac{1}{8}x^3 - y^6$
28. $a^3 + \frac{1}{64}b^3$
29. $\frac{1}{27}x^3 - \frac{1}{8}y^3$
30. $\frac{1}{32}a^5 - b^5$

Reduce to lowest terms:

31. $\dfrac{x^4 - y^4}{x^3 - y^3}$

32. $\dfrac{x^3 + 8}{x^4 - 16}$

33. $\dfrac{x^3 + y^3}{x^5 + y^5}$

34. $\dfrac{m^4 - 16}{m^5 - 32}$

35. $\dfrac{8\,c^3 - d^3}{16\,c^4 - d^4}$

36. $\dfrac{1 + 27\,x^6}{1 - 9\,x^4}$

Simplify:

37. $\dfrac{8\,x^3 - 27}{4\,x^2 - 9} \cdot \dfrac{4\,x^2 + 12\,x + 9}{4\,x^2 + 6\,x + 9} \cdot \dfrac{2\,x - 3}{6\,x^2 + 5\,x - 6}$

38. $\dfrac{m^3 + n^3}{m^3 - n^3} \cdot \dfrac{m - n}{m + n} \div \dfrac{(m - n)^2 + mn}{(m + n)^2 - mn}$

39. $\dfrac{3\,m^4 - 3\,n^4}{4\,m^3 - 4\,n^3} \cdot \dfrac{2\,m^2 - 2\,n^2}{6\,m^2 + 6\,n^2} \div \dfrac{cm^2 + 2\,cmn + cn^2}{4\,m^2 + 4\,mn + 4\,n^2}$

Factor:

40. $(x - y)^3 + z^3$

41. $(r - s)^3 - 8\,t^3$

42. $x^3 + (x - 1)^3$

43. $a^3 - 8(a - b)^3$

44. $27\,a^3 + 8(a - b)^3$

45. $(x - y)^3 - (x + y)^3$

46. $(a + 2\,b)^3 + (a - 2\,b)^3$

47. $(3\,x - y)^3 - (3\,x + y)^3$

48. $(m - 2\,n)^3 - (m + 2\,n)^3$

49. $(x + 2\,y)^3 + (2\,x + y)^3$

SUPPLEMENTARY TOPICS

198. The **Remainder theorem** makes it possible to find the remainder in certain division problems by a short process.

It is known that *dividend* = *divisor* × *quotient* + *remainder.*

Suppose that $x^3 + x^2 - 2$ is divided by $x - 2$; then:

$x^3 + x^2 - 2 = (x - 2) \cdot Q + R.$

Let $x = 2$; then:

$8 + 4 - 2 = 0 \cdot Q + R.$

$\therefore\ 10 = 0 \cdot Q + R.$

$\therefore\ 10 = R.$

That is, the remainder is 10.

Check. — See solution on right.

$$\begin{array}{r|l}
 & x^2 + 3\,x + 6 \\ \hline
x^3 + x^2 - 2 & x - 2 \\
x^3 - 2\,x^2 & \\ \hline
3\,x^2 & \\
3\,x^2 - 6\,x & \\ \hline
6\,x - 2 & \\
6\,x - 12 & \\ \hline
10 = R. &
\end{array}$$

At the left, the correct remainder was obtained by substituting 2 for x in the given expression. This suggests the

Remainder theorem. — If a rational and integral polynomial involving x be divided by $x - a$, the remainder may be found by substituting a for x in the given polynomial.

NOTE. — A polynomial is **integral** if it does not have any negative powers of the variable x, or any fractions with x in the denominator.

A polynomial is **rational** if it does not have any fractional powers of x, or any x's under radical signs.

Proof. — 1. The polynomial (containing x) $= (x - a) \cdot Q + R$.

2. $\therefore$ The polynomial (x replaced by a) $= (a - a) \cdot Q + R$.

3. $\therefore$ The polynomial (x replaced by a) $= 0 \cdot Q + R$, or R.

Example 1. — Find the remainder when $x^4 - 3x + 5$ is divided by $x - 3$.

Solution. — 1. Comparing $x - a$ and $x - 3$, a must be 3.

2. Substituting 3 for x in $x^4 - 3x + 5$,

$$R = 3^4 - 3 \cdot 3 + 5. \quad \therefore R = 81 - 9 + 5, \text{ or } 77.$$

Example 2. — Find the remainder when the divisor is $x + 3$.

Solution. — 1. Comparing $x - a$ and $x + 3$, a must be -3.

2. Substituting -3 for x in $x^4 - 3x + 5$,

$$R = (-3)^4 - 3 \cdot (-3) + 5. \quad \therefore R = 81 + 9 + 5, \text{ or } 95.$$

EXERCISE 147

Find the remainder when:

1. $x^4 - 2x^3 + 5$ is divided: *a.* by $x - 1$; *b.* by $x + 2$.

2. $3x^3 - 2x^2 - 4x + 1$ is divided: *a.* by $x - 2$; *b.* by $x + 3$.

3. $y^4 - 3y^2 - 7$ is divided: *a.* by $y + 2$; *b.* by $y - 1$.

4. $z^5 - 4z^3 + z - 30$ is divided: *a.* by $z - 2$; *b.* by $z + 1$.

5. $3m^4 - 5m^3 - m - 6$ is divided: *a.* by $m - 2$; *b.* by $m - 3$.

199. Synthetic division is a short process for finding the *quotient* as well as the remainder when a polynomial containing x is divided by a binomial of the form $x - a$.

Consider the two solutions:

Solution a

$$\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & 5x^2 & +\ 4x & +\ 6 & \\ \cline{2-5}
x-3 & 5x^3 & -\ 11x^2 & -\ 6x & -\ 10 \\
\multicolumn{1}{r}{} & 5x^3 & -\ 15x^2 & & \\ \cline{2-3}
\multicolumn{1}{r}{} & & 4x^2 & -\ 6x & \\
\multicolumn{1}{r}{} & & 4x^2 & -\ 12x & \\ \cline{3-4}
\multicolumn{1}{r}{} & & & 6x & -\ 10 \\
\multicolumn{1}{r}{} & & & 6x & -\ 18 \\ \cline{4-5}
\multicolumn{1}{r}{} & & & & +\ 8
\end{array}$$

Solution b

$$\begin{array}{r|rrrr}
x+3 & 5x^3 & -\ 11x^2 & -\ 6x & -\ 10 \\
 & 5 & +\ 15 & +\ 12 & +\ 18 \\ \cline{2-5}
\multicolumn{1}{r}{} & 5 & +\ 4 & +\ 6\ \| & +\ 8
\end{array}$$

Quotient: $5x^2 + 4x + 6$.

Remainder: $+8$.

The method of performing the solution *b*:

1. -3 of the original divisor is changed to $+3$.
2. $5x^3 \div x = 5x^2$. Place 5 in the third line.
3. $+3 \cdot +5 = +15$. Add the product, $+15$, to -11. Place the sum, $+4$, in the third line.
4. $+3 \cdot +4 = +12$. Add the product, $+12$, to -6. Place the sum, $+6$, in the third line.
5. $+3 \cdot +6 = +18$. Add the product, $+18$, to -10. Place the sum, $+8$, in the third line.
6. The numbers 5, $+4$, and $+6$ are the coefficients of the quotient. Since $5x^3 \div x = 5x^2$, the full quotient is $5x^2 + 4x + 6$. The last number of the third line, $+8$, is the remainder.

A partial explanation follows:

1. In Step 1, -3 is changed to $+3$. This permits *addition* in Steps 3, 4, and 5 instead of the customary *subtraction*. Thus, in solution *a*, when $-15x^2$ is *subtracted* from $-11x^2$, the result is $4x^2$; in solution *b*, when $+15x^2$ is *added* to $-11x^2$, the result is again $4x^2$.
2. In Step 3, $+4$ below the line represents, first, the coefficient of the first term of the remainder as in solution *a*. When $4x^2$ is divided by x the quotient is $+4x$, so that 4 may properly be considered also the coefficient of the second term of the quotient.

Example. — Divide $7x^4 - 29t^2x^2 - 3t^4$ by $x + 2t$.

Solution. — Change $x + 2t$ to $x - 2t$.

$$\begin{array}{r|rrrrr}
x-2t & 7x^4 & +\ 0tx^3 & -\ 29t^2x^2 & +\ 0t^3x & -\ 3t^4 \\
 & 7 & -\ 14t & +\ 28t^2 & +\ 2t^3 & -\ 4t^4 \\ \cline{2-6}
\multicolumn{1}{r}{} & 7 & -\ 14t & -\ t^2 & +\ 2t^3\ \| & -\ 7t^4
\end{array}$$

Quotient: $7x^3 - 14tx^2 - t^2x + 2t^3$. Remainder: $-7t^4$.

NOTE 1. — When powers of x are missing, supply them with coefficients zero, as in this example.

NOTE 2. — In §§ 198 and 199, two short processes for finding the remainder in certain division problems are given. Each is important. The second has the further advantage of determining the quotient as well.

EXERCISE 148

Divide by synthetic division:

1. $x^3 + 2x^2 - 2x + 5$ by $x - 1$
2. $y^3 - 4y^2 + y + 6$ by $y + 1$
3. $2x^3 - 3x^2 + x - 8$ by $x - 2$
4. $3z^4 + 4z^3 - 12z^2 + 8z + 6$ by $z + 3$
5. $x^4 + x^2 - 6$ by $x + 2$
6. $3t^4 - 8t^3 - 30$ by $t - 3$
7. $m^4 - 256$ by $m + 4$
8. $5r^3 + 6r^2 - r - 12$ by $r - 2$
9. $x^4 + 2mx^3 - 5m^2x^2 + 12m^4$ by $x - 2m$
10. $2y^5 - 15a^2y^3 + 11a^3y^2 - 21a^5$ by $y + 3a$

200. The factor theorem makes it possible to factor certain polynomials which are not factorable otherwise.

a. Consider $(x - 2)(x + 3) = x^2 - x - 6$.

Observe that the numbers 2 and 3, of the binomials, are factors of the last term 6, of the trinomial. If the polynomial to be factored is arranged in descending powers of one letter, then the last term always is the product of the last terms of the factors of the polynomial.

Thus, -12, of $x^3 + 3x^2 - 4x - 12$ is the product of the last terms of the factors of $x^3 + 3x^2 - 4x - 12$.

Therefore, the last terms of the factors must be selected from 1, 2, 3, 4, 6, and 12. For example, 5 cannot be the last term of any factor of $x^3 + 3x^2 - 4x - 12$.

b. Factor $x^3 + x^2 - 2$ by trial.

Solution. — 1. The factors must end in either 1 or 2.

2. Is $x - 1$ a factor? If it is, the remainder must be 0 when $x^3 + x^2 - 2$ is divided by $x - 1$.

By the remainder theorem, $R = 1 + 1 - 2 = 0$. (See page 299.)

3. $\therefore\ (x - 1)$ is a factor of $x^3 + x^2 - 2$.

Factor theorem. — If a rational and integral polynomial involving x becomes zero when x is replaced by a, then the polynomial has $x - a$ as a factor.

Proof. — 1. Considering the polynomial the dividend and $(x - a)$ the divisor, the remainder is the value obtained when a is substituted for x in the polynomial.

2. By the theorem, this value is zero, and $\therefore$ the remainder $= 0$.

3. $\therefore\ x - a$ is a factor.

When applying this theorem:

1. Determine mentally, if possible, some number a for which the polynomial becomes zero. (Factor theorem.)

2. Divide the polynomial by $(x - a)$ by synthetic division. This division will give further assurance that the remainder is zero, and will also determine the other factor, — the quotient. The quotient often may be factored.

Example. — Find the factors of $x^3 + 3x^2 - 4x - 12$.

Solution. — 1. When $x = 1$, then (mentally) $x^3 + 3x^2 - 4x - 12 = -12$. $\therefore\ x - 1$ is not a factor of the polynomial.

2. When $x = -1$, $x^3 + 3x^2 - 4x - 12 = -6$. $\therefore\ x + 1$ is not a factor.

3. When $x = 2$, $x^3 + 3x^2 - 4x - 12 = 0$. $\therefore\ x - 2$ is a factor.

4. Dividing by synthetic division:

$$\begin{array}{r|rrrr} x + 2 & x^3 & +\,3x^2 & -\ 4x & -\,12 \\ & 1 & +\,2 & +\,10 & +\,12 \\ \hline & 1 & +\,5 & +\ \ 6\,\| & 0 \end{array}$$

Remainder $= 0$. $\therefore\ x - 2$ is a factor. The other factor is $x^2 + 5x + 6$.

5. $\therefore\ x^3 + 3x^2 - 4x - 12 = (x - 2)(x^2 + 5x + 6)$

$$= (x - 2)(x + 2)(x + 3).$$

EXERCISE 149

Find the factors of:

1. $x^3 - 4x^2 + x + 6$

2. $y^3 + 2y^2 - 9y - 18$

3. $m^3 - 5m^2 + 3m + 9$

4. $r^3 + 3r^2 - 4r - 12$

5. $s^3 - 5s^2 - 2s + 24$

6. $2x^3 + 6x^2 + 3x - 2$

7. $3y^3 - 10y^2 + 2y + 3$

8. $r^3 - 3r^2 - 3r - 4$

9. $s^4 + s^3 - 3s^2 - 4s - 4$

10. $m^4 + 5m^3 + 5m^2 - 5m - 6$

11. Factor $x^3 + 4ax^2 - 3a^3$

Solution. — 1. Is $(x + a)$ a factor?

$$\begin{array}{r|rrrr} x - a & x^3 & +\,4ax^2 & +\,0\cdot a^2x & -\,3a^3 \\ & & -\,a & \text{etc.} & \text{etc.} \\ \cline{2-5} & 1 & +\,3a & \text{etc.} & \text{etc.} \end{array}$$

(Complete the solution.)

12. $x^3 - 4ax^2 + 5a^2x - 2a^3$

13. $x^3 - 4tx^2 + t^2x + 6t^3$

14. $x^3 - 4cx^2 - 15c^2x + 18c^3$

15. $x^3 - 5mx^2 - 8m^2x + 48m^3$

EXERCISE 150

Miscellaneous Examples in Factoring, Fractions, and Equations Solved by Factoring

Factor:

1. $a^2bc - ac^2d + ab^2d - bcd^2$

2. $4ax + 5by - 4ay - 5bx$

3. $4a^2 + 6b - 9 - b^2$

4. $16c^2 - 4d^2 + 20rd - 25r^2$

5. $3a^3 - 21a^2 - a + 7$

6. $x^2 - y^2 - z^2 + 2yz$

7. $y^3 - y^2 - y - 2$

8. $x^{10} - r^{10}$

9. $x^6 + 2x^3 + 1$

10. $4a^3 + 15 - 6a - 10a^2$

11. $x^3 - 3x^2 - 10x + 24$

12. $5m^5n - 5mn^5$

13. $c^3 + 2c^2 - 9c - 18$

14. $6t^3 - 9t + 8t^2 - 12$

15. $m^3 - 13m - 12$

16. $x^4 - 2x^2y^2 + y^4$

17. $x^4 - x^3 - 5x^2 - x - 6$

18. $ax^3 + bx^2 - ac^2x - bc^2$

19. $a^3 - 19a - 30$

20. $5rx + 5ry - sx - sy$

21. $a^6 - a^4 - a^2 + 1$

22. $r^3 + 2 - r^2 - 3r$

23. $a^3b^3 - 8b^3 - 8y^3 + a^3y^3$

24. $2c^4 - 4c^3 - 3c^2 + 6c$

25. $(a^2 + 2a)^2 - 11(a^2 + 2a) - 24$

26. $x^4 + 2x^2y^2 + 9y^4$

27. $x^6 + 7x^3 - 8$

28. $a^3 + 2a^2b - 5ab^2 - 6b^3$

29. $4m^3 - 4m^2x + 7mx^2 - 7x^3$

30. $\frac{1}{128} - a^7$

Reduce to lowest terms:

31. $\dfrac{x^2 - y^2 - 2yz - z^2}{x^2 + 2xy + y^2 - z^2}$

32. $\dfrac{a^3 - 3a^2 + a - 3}{a^3 - 2a^2 + a - 2}$

33. $\dfrac{c^3 - 3c^2d + cd^2 - 3d^3}{c^2 - 9d^2}$

34. $\dfrac{x^3 + 2x^2y - xy^2 - 2y^3}{x^2 - 4y^2}$

Find:

35. $\dfrac{a^2 - 2ab + b^2 - c^2}{a^2 - b^2 - 2bc - c^2} \cdot \dfrac{a^2 + 2ab + b^2 - c^2}{a^2 - b^2 + 2bc - c^2}$

36. $\dfrac{ax - ay + by - bx}{x^2 - y^2} \cdot \dfrac{ax + ay - bx - by}{a^2 - 2ab + b^2}$

37. $\dfrac{m^3 + 2m^2n - mn^2 - 2n^3}{m^2 - 4n^2} \div \dfrac{m^4 - 2m^2n^2 + n^4}{m^3 - 2m^2n + mn^2 - 2n^3}$

38. $\dfrac{x^2 - 3x + 2}{3x^3 + 6x^2 + x + 2} \div \dfrac{x^3 - x^2 + 2x - 2}{3x^4 + 7x^2 + 2}$

39. $\dfrac{2rt - 2rn - 3st + 3sn}{rt - rn - sn + st} \cdot \dfrac{r^3 - rs^2 + r^2s - s^3}{2r^2 - rs - 3s^2}$

40. $\left(1 + \dfrac{ab - c^2}{a^2 + ab + b^2}\right) \div \left(1 - \dfrac{3ab + c^2}{a^2 + ab + b^2}\right)$

41. Solve the equation $x^2 + ax - 3ab - 3bx = 0$.

Solution. — 1. $x^2 + ax - 3ab - 3bx = 0.$

2. Factoring, $x(x + a) - 3b(x + a) = 0.$

3. $\therefore\ (x - 3b)(x + a) = 0.$

4. $\therefore\ x_1 = 3b$, and $x_2 = -a$. (See page 147)

Check. — When $x = 3b$, does $(3b)^2 + a(3b) - 3ab - 3b(3b) = 0$?
Does $9b^2 + 3ab - 3ab - 9b^2 = 0$? Yes.
Similarly for the root $x = -a$.

Solve the following equations for x:

42. $x^2 - ax + bx - ab = 0$

43. $6x^2 + 2nx - 3mx - mn = 0$

44. $x^3 - 4x + 3x^2 - 12 = 0$

45. $x^3 - 3x - 2x^2 + 6 = 0$

46. $ax^2 - acx - bx + bc = 0$

47. $abx^2 - 2bx + 3ax - 6 = 0$

48. $x^2 - r^2 - 2rs - s^2 = 0$

49. $x^2 - m^2 + 2mn - n^2 = 0$

50. $x^2 - bx - cx - ax + ab + ac = 0$

51. $a^2 - ax = 2x + a + 6$

52. $acx^2 + bcx - abx - b^2 = 0$

53. $2mnx^2 + nx - 2mx - 1 = 0$

54. $a^2x^2 - b^2 + 2bc - c^2 = 0$

55. $cdmx^2 - amx - cdnx + an = 0$

56. $\dfrac{x^2 - x + 1 - \dfrac{2}{x+1}}{x + \dfrac{1}{x+1}}$

58. $\dfrac{\left(\dfrac{1}{y} + \dfrac{2}{x}\right)^2 - \dfrac{6}{xy}}{\dfrac{x}{y^2} + \dfrac{8y}{x^2}}$

57. $\dfrac{\dfrac{9x^2}{y^2} - \dfrac{8y}{3x}}{\dfrac{3x}{2y} + 1 + \dfrac{2y}{3x}}$

59. $\dfrac{16m^2 - \dfrac{1}{m^2}}{\left(1 + \dfrac{1}{4m^2}\right)^2 - \dfrac{1}{m^2}}$

60. $\left(1 - \dfrac{xy}{x^2 - xy + y^2}\right)\left(1 - \dfrac{xy}{x^2 + 2xy + y^2}\right) \div \dfrac{x^3 - y^3}{x^3 + y^3}$

201. Equations of higher degree solved by factoring. A first-degree equation, page 104 (having one unknown), has one root; a second-degree equation has two roots, page 148.

It is proved in higher courses in algebra that an equation of degree n has n roots.

The roots of equations of degree higher than the second are not obtained readily except in special equations which can be solved partially at least by the factoring method.

Example 1. — Find the roots of $x^4 - 13x^2 + 36 = 0$.

Solution. — 1. Factoring, $(x^2 - 4)(x^2 - 9) = 0$.

2. $\therefore\ x^2 - 4 = 0;\quad \therefore\ x^2 = 4;\quad \therefore\ x = 2$, or $x = -2$.
Also, $x^2 - 9 = 0;\quad \therefore\ x^2 = 9;\quad \therefore\ x = 3$, or $x = -3$.

Example 2. — Solve the equation $y^3 + 2y^2 - 4y + 1 = 0$.

Solution. — 1. Factoring by the factor theorem,

$$(y - 1)(y^2 + 3y - 1) = 0.$$

2. $\therefore\ y - 1 = 0$, or $y = 1$.

3. Also, $y^2 + 3y - 1 = 0;\ \therefore\ y = \dfrac{-3 \pm \sqrt{9 + 4}}{2}$, or $\dfrac{-3 \pm \sqrt{13}}{2}$.

4. $\therefore\ y = \dfrac{-3 \pm 3.605}{2};\ y = .302^+$, or -3.302^+.

5. Hence, $y_1 = 1;\ y_2 = .302^+;\ y_3 = -3.302^+$.

EXERCISE 151

Solve the following equations:

1. $x^4 - 5x^2 + 4 = 0$
2. $y^4 - 29y^2 + 100 = 0$
3. $z^4 - 8z^2 - 9 = 0$
4. $w^4 - 5w^2 - 36 = 0$
5. $t^4 + 12t^2 - 64 = 0$
6. $4x^4 - 13x^2 + 9 = 0$
7. $4y^4 - 19y^2 + 12 = 0$
8. $9a^4 - 35a^2 - 4 = 0$
9. $6x^4 - 5x^2 + 1 = 0$
10. $2x^4 + 11x^2 + 5 = 0$
11. $x^3 - x^2 - 5x + 5 = 0$
12. $x^3 - 2x^2 - x + 2 = 0$
13. $y^3 - 2y^2 - 6y - 3 = 0$
14. $2y^3 + y^2 - 6y + 3 = 0$
15. $c^3 - 6c^2 + 11c - 6 = 0$
16. $2c^3 - 5c^2 + c + 2 = 0$
17. $6t^3 - 11t^2 - 4t + 4 = 0$
18. $t^4 + t^3 - 2t^2 - t + 1 = 0$
19. $a^4 - 5a^3 + 5a^2 + 5a - 6 = 0$

20. $x^4 - x^3 - 7x^2 + x + 6 = 0$

21. $x^4 - x^3 - 4x^2 + 2x + 4 = 0$

22. $y^3 + 3y^2 - 4 = 0$

23. $x^4 - x^3 - 3x^2 - 3x - 18 = 0$

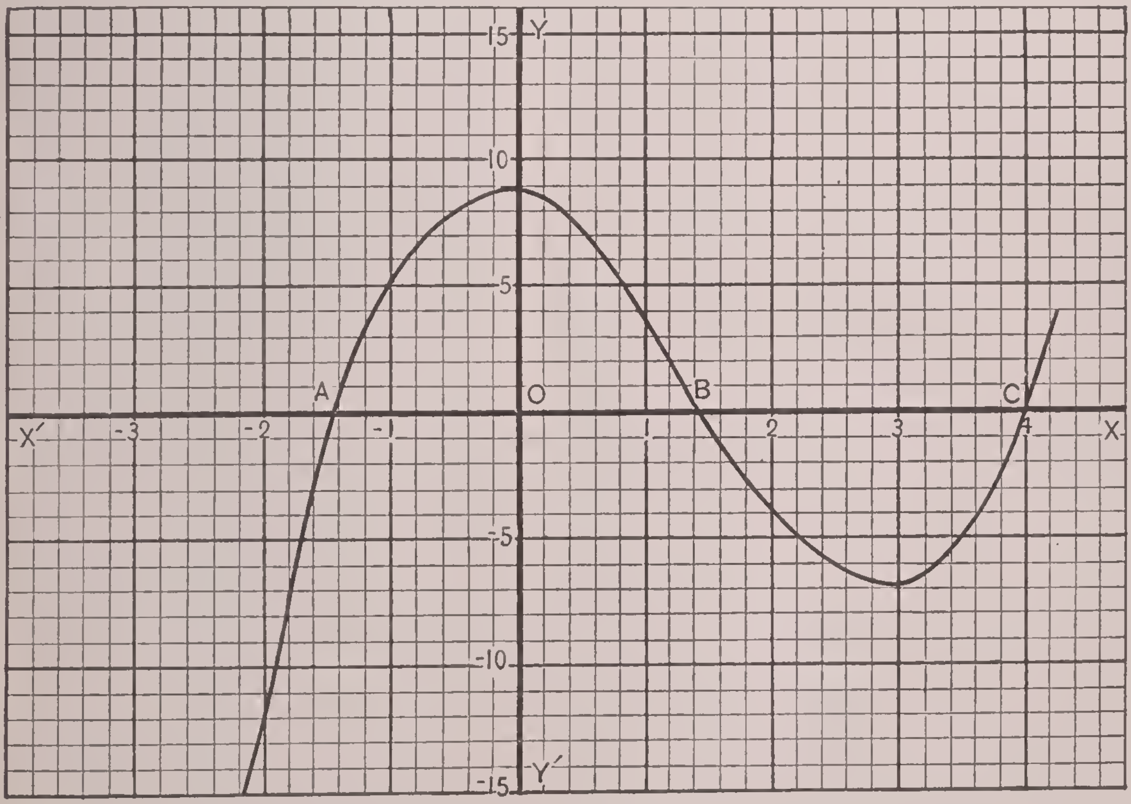
24. $6t^4 + 5t^3 - 5t^2 - 5t - 1 = 0$

202. Graphical solution of equations. In § 201, the statement was made that an equation of the nth degree, having one unknown, has n roots, but that these roots are not readily found for equations of degree above the second. Such equations may be solved graphically as in § 188.

Example. — Solve the equation $x^3 - 4x^2 - 2x + 8 = 0$.

Solution. — 1. Let $y = x^3 - 4x^2 - 2x + 8$.

When x =	0	1	2	3	4	5	− 1	− 2	− 3
then y =	8	3	− 4	− 7	0	23	5	− 12	− 49



2. The curve crosses the horizontal axis at points A, B, and C. Hence its roots are, approximately, − 1.42, + 1.42, and + 4.

Check. — This equation can be solved as in § 201. Using the factor theorem: $x^3 - 4x^2 - 2x + 8 = (x - 4)(x^2 - 2) = 0$.

$$\therefore\ x = 4;\ \text{also } x^2 = 2, \text{ or } x = \pm\sqrt{2} = \pm 1.414.$$

Clearly the results ± 1.42 obtained graphically are close to the roots ± 1.414.

EXERCISE 152

Solve the following equations graphically:

1. $x^3 - 3x^2 - x + 3 = 0$
2. $x^3 - 4x^2 - 7x + 10 = 0$
3. $x^3 + x^2 - 6x = 0$
4. $x^4 - 10x^2 + 9 = 0$
5. $x^3 + x^2 - 10x - 10 = 0$
6. $x^3 - x^2 - 8x + 8 = 0$
7. $x^3 - 4x^2 - x + 4 = 0$
8. $x^3 + x^2 - 5x + 3 = 0$
9. $x^3 + 2x^2 + x + 2 = 0$
10. $x^4 - x^3 - 8x^2 + 12x = 0$

XVI. QUADRATIC EQUATIONS HAVING TWO VARIABLES. GRAPHICAL SOLUTION

203. Graph of a single equation.

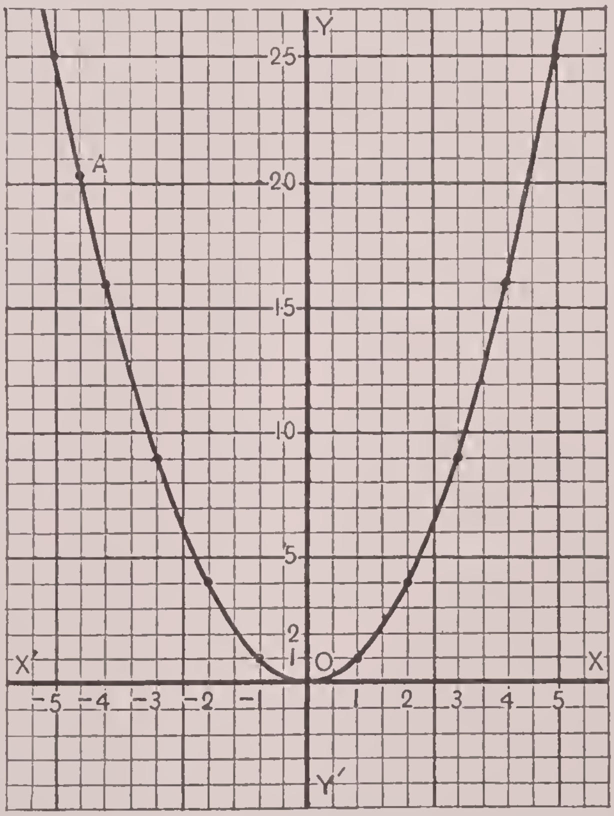
Example 1. — Draw the graph of $y - x^2 = 0$.

Solution. — 1. Solve the equation for y in terms of x.

$$y = x^2$$

2.

If x =	0	1	2	3	4	5	-1	-2	-3	-4	-5
then y =	0	1	4	9	16	25	$+1$	$+4$	$+9$	$+16$	$+25$



3. The graph obtained is a **parabola**.

4. The coördinates of any point on the graph satisfy the equation.

Thus, $A = (-4.5, 20^+)$.

Substituting in $y = x^2$, does $(20^+) - (-4.5)^2 = 0$?

Does $(20^+) - (20.25) = 0$? Approximately, yes.

5. The coördinates of any point not on the graph do not satisfy the equation of the graph. *E.g.*, you will find that the coördinates of the point (3, 15) do not satisfy the equation.

Example 2. — Draw the graph of $x^2 + y^2 = 25$.

Solution. — 1. Solve the equation for y. $y = \pm\sqrt{25 - x^2}$.

2.

When x =	0	$+1$	$+2$	$+3$	$+4$	$+5$
then y =	$\pm\sqrt{25}$ ± 5	$\pm\sqrt{24}$ ± 4.8	$\pm\sqrt{21}$ ± 4.5	$\pm\sqrt{16}$ ± 4	$\pm\sqrt{9}$ ± 3	$\pm\sqrt{0}$ 0

3. When x is -3, $y = \pm\sqrt{25-(-3)^2} = \pm\sqrt{25-9} = \pm 4$. Similarly, when $x = -1, -2, -4$, and -5.

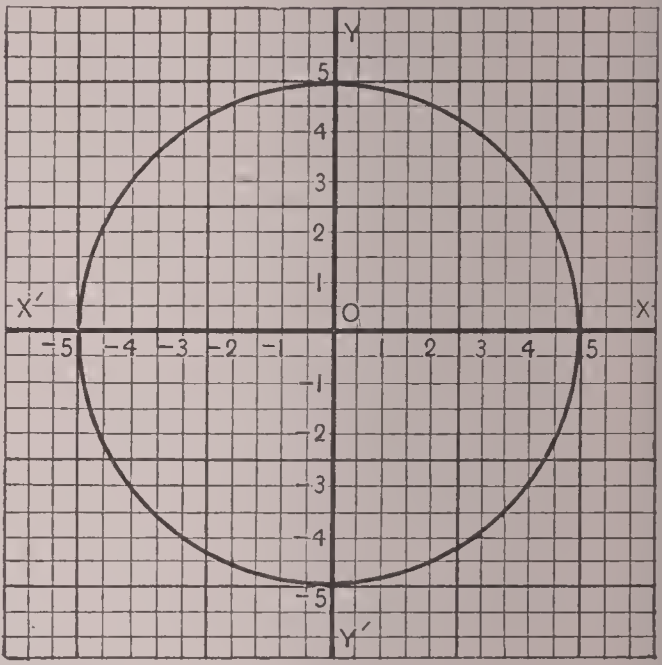
4. y has two values for each value of x. Thus, when $x = +4$, $y = +3$, or -3. Therefore $(4, 3)$ and $(4, -3)$ are points on the graph.

5. When x is greater than 5, y is imaginary. Thus, when $x = 6$, $y = \pm\sqrt{25-36}$ or $\pm\sqrt{-11}$. This means that there are not any points on the graph for values of x greater than $+5$, or less than -5.

6. The graph appears at the right.

It is a **circle**.

7. Every equation of the form $x^2 + y^2 = r^2$ is a circle with its center at the origin, and its radius equal to r.



Example 3. — Draw the graph of $9x^2 + 25y^2 = 225$.

Solution. — 1. $y^2 = \dfrac{225 - 9x^2}{25}$. $\therefore\ y = \pm\frac{1}{5}\sqrt{225 - 9x^2}$.

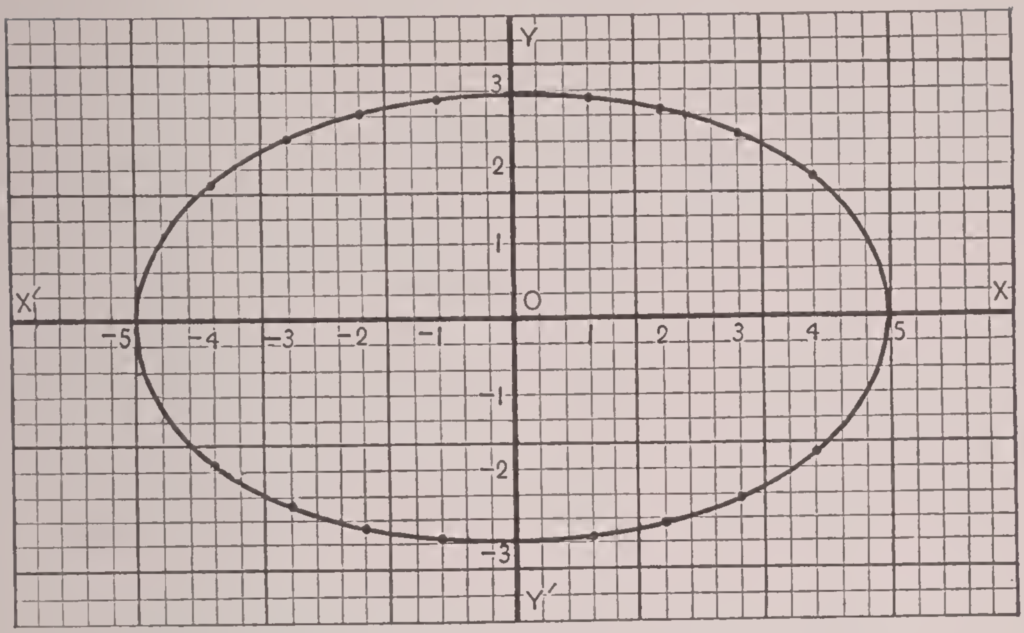
2. When $x = 2$, $y = \pm\frac{1}{5}\sqrt{225-36} = \pm\frac{1}{5}\sqrt{189} = \pm\frac{1}{5}(3\sqrt{21})$
$= \pm\frac{3}{5}(4.5) = \pm 2.7$. Similarly,

When $x =$	0	$+1$	$+2$	$+3$	$+4$	$+5$	$+6$
then $y =$	$\pm\dfrac{\sqrt{225}}{5}$	$\pm\frac{1}{5}\sqrt{216}$	$\pm\frac{1}{5}\sqrt{189}$	$\pm\frac{1}{5}\sqrt{144}$	$\pm\frac{1}{5}\sqrt{81}$	$\sqrt{0}$	$\pm\frac{1}{5}\sqrt{-99}$
$=$	$\pm\frac{15}{5}$	$\pm\frac{1}{5}(14.6)$	$\pm\frac{1}{5}(13.5)$	$\pm\frac{12}{5}$	$\pm\frac{9}{5}$	0	imag'y
$=$	± 3	± 2.9	± 2.7	± 2.4	± 1.8	0	imag'y

3. For $x = -3$, y has the same values that it has for $x = +3$. Similarly when $x = -1, -2, -4$, and -5.

4. For each value of x there are two values of y. The graph follows.

5. The graph is an **ellipse**. Every equation of the form $ax^2 + by^2 = c$, in which a, b, and c are positive, and a does not equal b, has for its graph an ellipse.

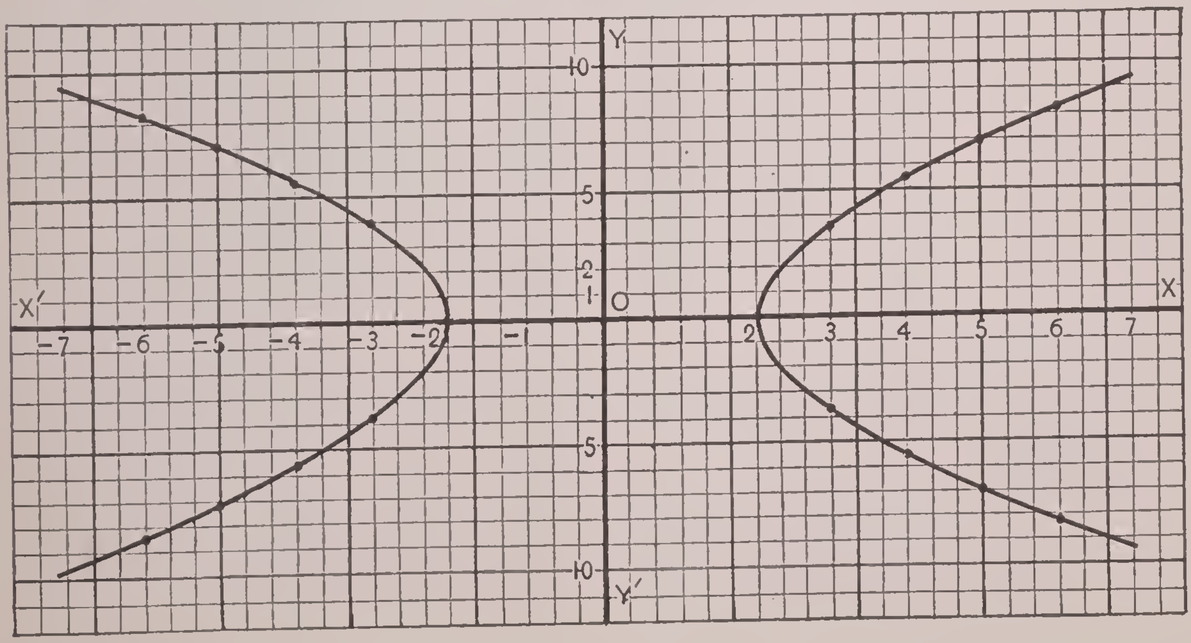


Example 4. — Draw the graph of $9x^2 - 4y^2 = 36$.

Solution. — 1. $y^2 = \dfrac{9x^2 - 36}{4}$; $\therefore\ y = \pm\frac{3}{2}\sqrt{x^2 - 4}$.

2. When $x = 1$, $y = \pm\frac{3}{2}\sqrt{1 - 4} = \pm\frac{3}{2}\sqrt{-3}$; $\therefore\ y$ is imaginary.

When $x =$	0	$+1$	$+2$	$+3$	$+4$	$+5$	$+6$
then $y =$	$\pm\frac{3}{2}\sqrt{-4}$ imag'y	$\pm\frac{3}{2}\sqrt{-3}$ imag'y	$\pm\frac{3}{2}\sqrt{0}$ 0	$\pm\frac{3}{2}\sqrt{5}$ ± 3.3	$\pm\frac{3}{2}\sqrt{12}$ ± 5.1	$\pm\frac{3}{2}\sqrt{21}$ ± 6.8	$\pm\frac{3}{2}\sqrt{32}$ ± 8.4



3. When $x = -1$, $y = \pm \frac{3}{2}\sqrt{(-1)^2 - 4} = \pm \frac{3}{2}\sqrt{1 - 4} = \pm \frac{3}{2}\sqrt{-3}$. These are the same values obtained when $x = +1$. Similarly for $x = -2, -3, -4$, etc. The graph follows.

4. This graph is a **hyperbola**. Every equation of the form $ax^2 - by^2 = c$, where a, b, and c are positive members, is a hyperbola.

EXERCISE 153

Draw the graphs of the following equations. Name the curves obtained.

1. $x^2 + y^2 = 49$

2. $y = 6x^2$

3. $x = 4y^2$

4. $x^2 + 9y^2 = 81$

5. $x^2 - 9y^2 = 16$

6. $x^2 + y^2 = 70$

7. $xy = 4$

8. $xy = -4$

9. $4x^2 + y^2 = 36$

10. $9x^2 + 4y^2 = 36$

11. Recalling the remark in Step 7, page 310, draw the following graphs, without any computation.

a. $x^2 + y^2 = 36$ *b.* $x^2 + y^2 = 81$ *c.* $x^2 + y^2 = 121$

SUPPLEMENTARY EXAMPLES

12. Draw the graph of $x^2 - xy + y^2 = 63$.

Solution. — 1. Another method of obtaining points on a graph follows:

Let $x = 0$; then $y^2 = 63$, or $y = \pm\sqrt{63}$, or ± 7.9.

$\therefore$ $x = 0$, $y = +7.9$ is a 1st point on the graph, and $x = 0$, $y = -7.9$ is a 2d point.

2. Let $x = -1$. $\therefore$ $1 + y + y^2 = 63$, or $y^2 + y - 62 = 0$.

$$\therefore\ y = \frac{-1 \pm \sqrt{1 + 248}}{2} = \frac{-1 \pm \sqrt{249}}{2} = \frac{-1 \pm 15.7}{2}.$$

$$\therefore\ y_1 = \frac{14.7}{2} \text{ or } 7.3; \quad \text{and } y_2 = \frac{-16.7}{2} = -8.8.$$

That is, $x = -1$, $y = 7.3$ is a 3d point, and $x = -1$, $y = -8.8$ is a 4th point on the graph.

In this manner, complete the table below and draw the graph.

When x =	0	-1	-3	-6	-9	$+3$	$+6$	$+9$
then y_1 =	$+7.9$	$+7.3$						
and y_2 =	-7.9	-8.8						

13. $y^2 + x^2 - 6x - 16 = 0$ **15.** $x^2 + y^2 - 8x - 4y = 5$

14. $y^2 + 4y + x^2 - 32 = 0$ **16.** $y^2 - xy - x^2 = 19$

17. On the same set of axes draw the graph of $xy = 5$, and of $xy = -5$. Notice the effect of changing the sign of 5.

18. Draw on one set of axes $y = x^2$ and $y = -x^2$.

19. Draw on one set of axes $x^2 + y^2 = 25$ and $x^2 - y^2 = 25$.

20. Draw on one set of axes:

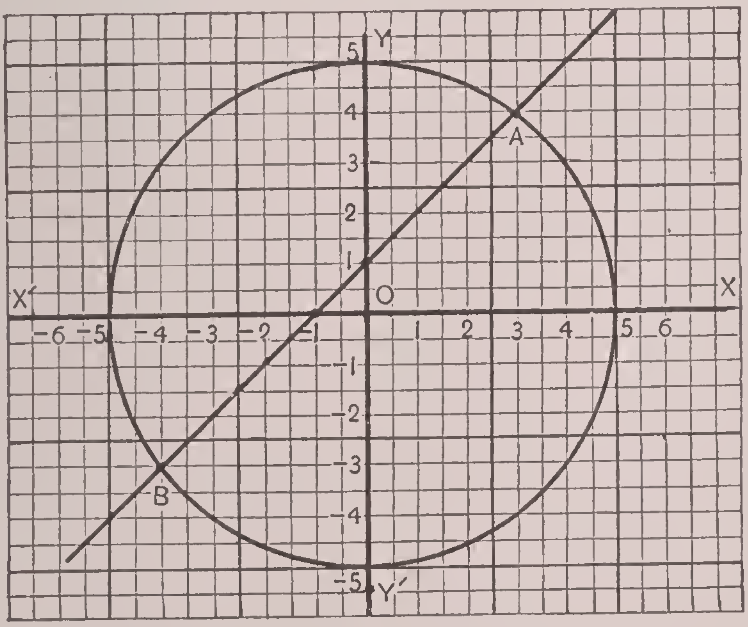
a. $x^2 + y^2 = 100$; *b.* $x^2 + 4y^2 = 100$, and *c.* $4x^2 + y^2 = 100$.

204. Solution of simultaneous quadratic equations graphically.

Example 1. — Solve the system $\begin{cases} x^2 + y^2 = 25 & (1) \\ x - y + 1 = 0 & (2) \end{cases}$

Solution. — 1. In Example 2, page 310, the graph of $x^2 + y^2 = 25$ is found to be the circle of radius 5, with center at the origin.

2. For equation 2: when $x = 0$, $-y + 1 = 0$, or $y = 1$. When $x = 2$, $2 - y + 1 = 0$, or $y = 3$. The graph is the straight line below.



3. A is on both graphs. Its coördinates are (3, 4). By substitution, you will find $x = 3$, $y = 4$ satisfy both equations.

4. Similarly the coördinates $(-4, -3)$ of B satisfy both equations.

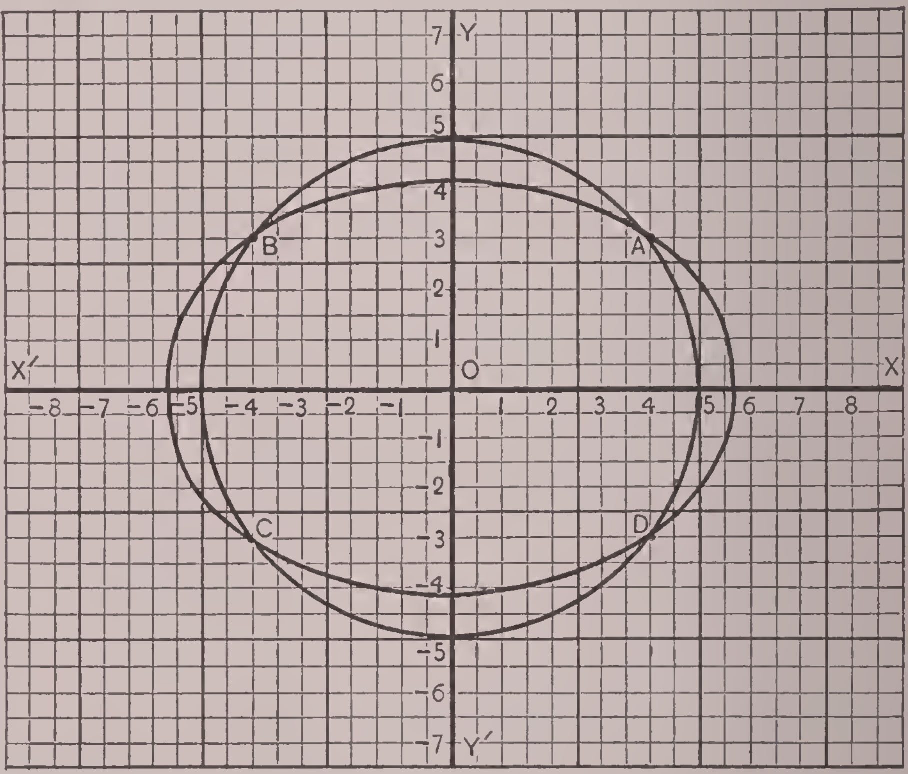
Note. — The graph of every linear equation having two variables is a straight line. (Rule, page 218.) The graph of every quadratic equation having two variables is one of the four curves discussed in § 203.

Since a straight line will, in general, cut such a curve in two points, **a system consisting of a linear equation and a quadratic equation will have two common solutions.**

If the straight line touches the curve at only one point (is tangent to the curve), the system has one "double" solution. If the straight line fails to touch the curve, the system has two imaginary solutions.

Example 2. — Solve the system $\begin{cases} x^2 + y^2 = 25 & (1) \\ x^2 + 2y^2 = 34 & (2) \end{cases}$

Solution. — 1. The graph of equation (1) is the circle obtained on page 310. The graph of the equation (2) is the ellipse in the figure below. (It is obtained as on page 310.)

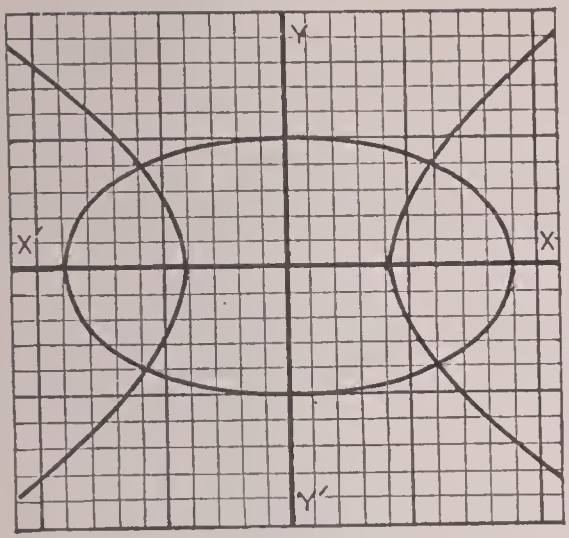


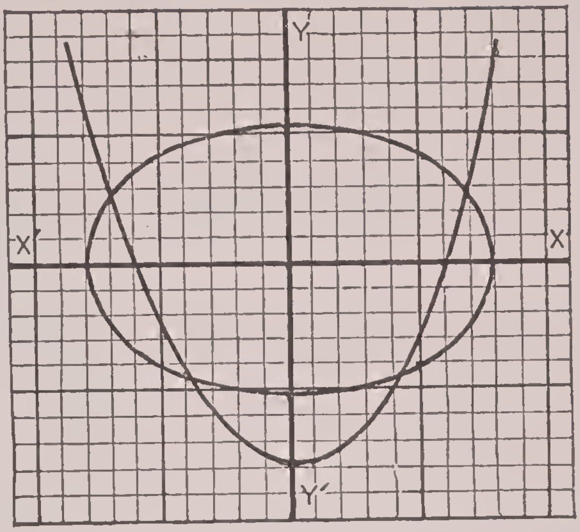
2. The common points on the graphs are:

A. $\begin{cases} x = 4 \\ y = 3 \end{cases}$ B. $\begin{cases} x = -4 \\ y = 3 \end{cases}$ C. $\begin{cases} x = -4 \\ y = -3 \end{cases}$ D. $\begin{cases} x = 4 \\ y = -3 \end{cases}$

By substituting these values in equations (1) and (2), you will find that these are common solutions of the given system of equations.

Generally, **two quadratic equations have four common solutions**, just as do these two equations. This becomes clear, if you draw upon one set of axes any two of the four curves found in § 203. For example,





However there are other possibilities. For example, the ellipse may intersect only one branch of the hyperbola. Then the system has two real solutions and two imaginary solutions.

EXERCISE 154

Solve the following systems graphically:

1. $\begin{cases} x^2 + y^2 = 52 \\ x - y = 2 \end{cases}$

2. $\begin{cases} 4x^2 + y^2 = 40 \\ x + y = 5 \end{cases}$

3. $\begin{cases} x^2 + y^2 = 34 \\ xy = 15 \end{cases}$

4. $\begin{cases} xy = 45 \\ y - x = 4 \end{cases}$

5. $\begin{cases} x^2 - 4y = 20 \\ x^2 + y^2 = 25 \end{cases}$

6. $\begin{cases} x^2 - y^2 = 19 \\ 2x + y^2 = 5 \end{cases}$

7. $\begin{cases} 4x^2 + 7y^2 = 32 \\ 11y^2 - 3x^2 = 41 \end{cases}$

8. $\begin{cases} x^2 - 4y^2 = 28 \\ 2x - 3y = 7 \end{cases}$

9. *a.* Draw the graph of $x^2 + y^2 = 25$.

b. On the same axes, draw the graph of $y = x^2$, and obtain the common solutions of it and $x^2 + y^2 = 25$.

c. Repeat part *b*, using $y = x^2 + 5$.

d. Repeat part *b*, using $y = x^2 - 8$.

e. Repeat part *b*, using $y = x^2 + 8$.

XVII. SYSTEMS OF EQUATIONS INVOLVING QUADRATICS

205. Review the definitions of independent equations (§ 150); simultaneous equations (§ 150); system of equations (§ 150); and dependent equations (§ 155).

206. Solving a system consisting of one linear and one quadratic equation.

Example. — Solve the system $\begin{cases} 3c + 2d = -2 & (1) \\ cd + 8c = 4 & (2) \end{cases}$

Solution. — 1. From (1), $c = \dfrac{-2 - 2d}{3}$. (3)

2. Substituting in (2), $d\left(\dfrac{-2-2d}{3}\right) + 8\left(\dfrac{-2-2d}{3}\right) = 4.$ (4)

3. Simplifying, $-2d - 2d^2 - 16 - 16d - 12 = 0.$ (5)

4. $\therefore\ 2d^2 + 18d + 28 = 0$, or $d^2 + 9d + 14 = 0$.

5. Solving for d, $d = -2$, or $d = -7$.

6.

When $d = -2$,	When $d = -7$,
in (1), $3c - 4 = -2$	in (1), $3c - 14 = -2$
or $c = \frac{2}{3}$.	or $c = 4$.

7. $\therefore$ The solutions are *a.* $\begin{cases} c = \frac{2}{3} \\ d = -2 \end{cases}$ *b.* $\begin{cases} c = 4 \\ d = -7 \end{cases}$

These solutions check when substituted in (1) and (2).

A system consisting of one linear and one quadratic equation always has two solutions.

EXERCISE 155

Solve the following systems of equations:

1. $\begin{cases} x^2 + y^2 = 85 \\ x - y = 1 \end{cases}$

2. $\begin{cases} 4r^2 + t^2 = 25 \\ 2r + t = 7 \end{cases}$

3. $\begin{cases} 3a - 2b = 3 \\ 3a^2 - 4ab + 2b^2 = 27 \end{cases}$

4. $\begin{cases} x^2 - 3xy + y^2 = -11 \\ y - x = -1 \end{cases}$

5. $\begin{cases} xy = 28 \\ x + y = 11 \end{cases}$

6. $\begin{cases} a^2 + b^2 = 130 \\ a - b = 8 \end{cases}$

7. $\begin{cases} 2r + t = 7 \\ rt = 6 \end{cases}$

8. $\begin{cases} x^2 - xy - y^2 = 19 \\ x - y = 7 \end{cases}$

9. $\begin{cases} d - 2c = 8 \\ d^2 - 3cd = 22 \end{cases}$

10. $\begin{cases} xy = 18 \\ y - x = 3 \end{cases}$

11. $\begin{cases} y - 3x = 7 \\ x^2 + xy = 2 \end{cases}$

12. $\begin{cases} 4x^2 - xy = 2(x + y) \\ y - x = 1 \end{cases}$

13. $\begin{cases} x^2 + y^2 = a^2 \\ 2x - y = a \end{cases}$

14. $\begin{cases} 32m + 4n = 8(m^2 - n^2) \\ m - 1 = 2n \end{cases}$

15. $\begin{cases} x^2 + y^2 - 6y = 0 \\ y + 2x = 0 \end{cases}$

16. $\begin{cases} 4x^2 + y^2 = 40 \\ 2x + y = 8 \end{cases}$

17. $\begin{cases} x^2 + y^2 - x - 2y = 0 \\ x + 2y = 5 \end{cases}$

18. $\begin{cases} 2c - d = 1 \\ 4cd + d^2 - 3c^2 = 9c - 7d \end{cases}$

19. $\begin{cases} 2y + 6x = 15xy \\ 6x + 2y = 5 \end{cases}$

20. $\begin{cases} 4x^2 - 6xy + 9y^2 = 63 \\ 2x - 3y = -3 \end{cases}$

207. Quadratic equations homogeneous except for the constant term.

Examples. — $4x^2 - xy + 3y^2 = 5$; $\frac{2}{3}x^2 - \frac{3}{4}y^2 + 2xy = \frac{7}{5}$.

Observe:

a. The variables do not appear in a denominator. Therefore the equation is **integral.**

b. The variables do not appear under radical signs. Therefore the equation is **rational.**

c. The sum of the exponents in the terms which do contain the variables is 2. It is this characteristic which makes the equation **homogeneous of degree two.**

Some systems consisting of two quadratics, homogeneous except for the constant term, can be solved by eliminating the constant terms.

Example. — Solve the system $\begin{cases} a^2 + ab + b^2 = 63 & (1) \\ a^2 - b^2 = -27 & (2) \end{cases}$

Solution. — 1. To eliminate 63 and 27:

M_3 (1) $\quad 3a^2 + 3ab + 3b^2 = 189. \quad$ (3)

M_7 (2) $\quad 7a^2 - 7b^2 = -189. \quad$ (4)

(3) + (4) $\quad 10a^2 + 3ab - 4b^2 = 0. \quad$ (5)

2. Solve (5) for a in terms of b:

$$(5a + 4b)(2a - b) = 0. \quad \therefore 5a + 4b = 0; \text{ and } 2a - b = 0.$$

3. Solve the two systems below:

$\begin{cases} a^2 - b^2 = -27 \\ 5a + 4b = 0 \end{cases}$

$a = -\dfrac{4b}{5}.$

$\therefore \dfrac{16b^2}{25} - b^2 = -27.$

$\therefore 16b^2 - 25b^2 = -675.$

$\qquad -9b^2 = -675.$

$\therefore b^2 = 75.$

$\therefore b = \pm\sqrt{75}$, or $\pm 5\sqrt{3}.$

But $a = -\dfrac{4b}{5}.$

$\therefore a = -\frac{4}{5}(\pm 5\sqrt{3}).$

$\therefore a = \mp 4\sqrt{3}.$

$\therefore$ Two solutions are:

$A \begin{cases} a = -4\sqrt{3} \\ b = +5\sqrt{3} \end{cases} \quad B \begin{cases} a = +4\sqrt{3} \\ b = -5\sqrt{3} \end{cases}$

$\begin{cases} a^2 - b^2 = -27 \\ 2a - b = 0 \end{cases}$

$\therefore b = 2a.$

$\therefore a^2 - 4a^2 = -27.$

$\therefore \quad -3a^2 = -27.$

$\therefore a^2 = 9.$

$\therefore a = \pm 3.$

$\therefore b = 2(\pm 3)$, or $\pm 6.$

$\therefore$ Two more solutions are:

$C \begin{cases} a = +3 \\ b = +6 \end{cases} \quad D \begin{cases} a = -3 \\ b = -6 \end{cases}$

Check. — These solutions may be checked by substituting them in the original equations.

NOTE 1. — In the first column of Step 3, notice that the value $-4\sqrt{3}$ of a comes from the value $+5\sqrt{3}$ of b. Therefore they form a solution (Solution A).

NOTE 2. — If one equation does not have a constant term, solve it at once for one variable in terms of the other, as in Step 2 above.

A system consisting of two quadratics has four solutions, — of which some may be imaginary.

EXERCISE 156

Solve the following systems:

1. $\begin{cases} x^2 + 3xy = 28 \\ xy + 4y^2 = 8 \end{cases}$

2. $\begin{cases} x^2 + y^2 = 37 \\ xy = 6 \end{cases}$

3. $\begin{cases} m^2 + mn = 10 \\ mn + n^2 = 15 \end{cases}$

4. $\begin{cases} a^2 - 3ab + 2b^2 = 3 \\ 2a^2 + b^2 = 6 \end{cases}$

5. $\begin{cases} 3xy - 2y^2 = 0 \\ 2x^2 - xy = 2 \end{cases}$

6. $\begin{cases} 3x^2 - 5xy + 3y^2 = 15 \\ x^2 - xy = 4 \end{cases}$

7. $\begin{cases} 3cd + d^2 = -14 \\ c^2 - cd = 30 \end{cases}$

8. $\begin{cases} x^2 - xy = 4 \\ x^2 + y^2 = 10 \end{cases}$

9. $\begin{cases} 4x^2 + 3y^2 = 28 \\ 2x^2 + xy + y^2 = 8 \end{cases}$

10. $\begin{cases} 2rt - 2r^2 - t^2 = -2 \\ t^2 - r^2 = 3 \end{cases}$

11. $\begin{cases} 2x^2 - 3xy = 2 \\ 4x^2 + 9y^2 = 10 \end{cases}$

12. $\begin{cases} h^2 + 25r^2 = 125 \\ 5r^2 + rh = 15 \end{cases}$

208. The number of common solutions of a system of equations. In § 206, where one equation is linear and the other is quadratic, each system has two common solutions. In § 207, where both equations are quadratic, each system has four common solutions. The number agrees in both cases with the number determined by the following

Rule. — Two rational and integral equations having two variables, whose degrees are *m* and *n* respectively, have *mn* common solutions.

Thus, a system consisting of a cubic equation and a quadratic equation should have $3 \cdot 2$ or 6 common solutions. Some of these may be imaginary, and some may be repeated solutions. For example, $x = 2$; $y = 1$ may appear twice in the list.

209. Equivalent systems of equations are two different systems which have the same common solutions. In terms of graphs, the intersection points of one system are the same as those of the other system.

210. **Solution of systems which are reducible by division to equivalent systems whose equations are of lower degree.**

Example. — Solve the system $\begin{cases} x^4 - y^4 = 240 & (1) \\ x^2 + y^2 = 20 & (2) \end{cases}$

Solution. — 1. By the rule of § 208, it would appear that there should be $4 \cdot 2$ or 8 common solutions.

In this example, however, $x^4 - y^4$ can be divided by $x^2 + y^2$.

2. Dividing (1) by (2), $x^2 - y^2 = 12$ (3)

3. Now form the system $\begin{cases} x^2 - y^2 = 12 & (3) \\ x^2 + y^2 = 20 & (2) \end{cases}$

This system will have $2 \cdot 2$ or 4 common solutions.

4. (3) + (2): $2x^2 = 32$; $x^2 = 16$; $x = \pm 4$.

5. When $x = +4$, $16 + y^2 = 20$; $y^2 = 4$; $y = \pm 2$.

$\therefore$ $x = 4$, $y = 2$, and $x = 4$, $y = -2$ are two solutions.

6. When $x = -4$, $16 + y^2 = 20$; $y^2 = 4$; $y = \pm 2$.

$\therefore$ $x = -4$, $y = 2$, and $x = -4$, $y = -2$ are two more solutions.

Check all four solutions by substituting in equations (1) and (2).

NOTE. — Whenever possible, divide one equation of the given system by the other, member by member, and form a new system consisting of the quotient equation and the divisor equation.

EXERCISE 157

Solve the following systems of equations:

1. $\begin{cases} x^2 - y^2 = 14 \\ x + y = 7 \end{cases}$

2. $\begin{cases} x^4 - y^4 = 65 \\ x^2 + y^2 = 13 \end{cases}$

3. $\begin{cases} x(x - y) = 50 \\ y(x - y) = 25 \end{cases}$

4. $\begin{cases} 2x^2 + 11xy + 12y^2 = 63 \\ 2x + 3y = 9 \end{cases}$

5. $\begin{cases} x^2 + 2xy - 3y^2 = 17 \\ x + 3y = 17 \end{cases}$

6. $\begin{cases} a^2 - b^2 = 4a + 6b - 8 \\ a - b = 2 \end{cases}$

7. $\begin{cases} x^3 - y^3 = 218 \\ x^2 + xy + y^2 = 109 \end{cases}$

8. $\begin{cases} x^3 - x^2y = 54 \\ x - y = 6 \end{cases}$

9. $\begin{cases} x^3 - y^3 = 19a^3 \\ x^2 + xy + y^2 = 19a^2 \end{cases}$

10. $\begin{cases} x^3 - 8y^3 = 208 \\ x - 2y = 14 \end{cases}$

11. $\begin{cases} x^3 + y^3 = 35 \\ x + y = 5 \end{cases}$

12. $\begin{cases} x^3 - y^3 = 56\,x + 28\,y - 56 \\ x - y = 2\,x + y - 2 \end{cases}$

13. $\begin{cases} a^3 + b^3 = 144 \\ a + b = 4 \end{cases}$

14. $\begin{cases} x^2 + xy - 6\,y^2 = 21 \\ xy + 3\,y^2 = 84 \end{cases}$

15. $\begin{cases} a^4 + ab^3 = 84 \\ a^3b + b^4 = 28 \end{cases}$

16. $\begin{cases} x^2 - 2\,xy + 6\,x = 32 \\ xy - 2\,y^2 + 6\,y = 8 \end{cases}$

EXERCISE 158

1. Find two numbers whose sum is -9 and the sum of whose squares is 101.

2. The sum of the squares of two numbers is 1274 and the larger is 5 times the smaller. Find the numbers.

3. The product of two numbers is 5 and the sum of their squares is 26. Find the numbers.

4. Find two numbers whose sum is 10 and whose product is 9.

5. Find two numbers whose difference is 5 and whose product is 6.

6. The product of two numbers is 48 and the sum of their squares is 100. Find the numbers.

7. The sum of two numbers is 8; the square of the first number increased by the product of the two numbers is equal to 49 diminished by the square of the second number. Find the numbers.

8. The square of the length of the diagonal of a rectangle is 40 and the area of the rectangle is 12 square feet. Find the dimensions of the rectangle.

9. Find two numbers such that the square of the first diminished by the product of the two numbers shall be 15, and such that the difference between their squares shall be 21.

10. Find two numbers such that their product increased by the first number shall be 8, and such that their product increased by twice the second number shall be 12.

11. In 6 hours, two boys row 16 miles downstream and back again. Their rate upstream is twice the rate of the current. Find the rate at which they row in still water and the rate of the current.

12. The sum of the squares of the two digits of a number is 74. If 18 be added to the number, the digits of the sum are the digits of the original number in reverse order. Find the number.

13. Find the sides of a rectangle if the area of the rectangle increased by the shorter side is 15 and if the area increased by the longer side is 16.

14. The sum of the volumes of two cubes is 72; and an edge of one cube plus an edge of the other cube is 6. Find the length of the edges of each cube.

15. Find the number of two digits such that if the digits be reversed, the new number minus the original number is 18, and their product is 403.

16. Find two numbers the sum of whose squares increased by the product of the numbers is 28, and the difference of whose cubes is 56.

17. The area of a trapezoid is 36 square inches. Its altitude is 3 inches. Find the two bases of the trapezoid if the square of the length of the lower base is 192 more than the square of the length of the upper base.

18. The sum of a certain fraction and its reciprocal is $\frac{10}{3}$. If three times the numerator is decreased by twice the denominator the result is equal to -12. Find the fraction.

19. In an isosceles triangle, the vertex angle is 60° more than the square of either base angle. Find the angles of the triangle.

20. The sum of the squares of the two digits of a number is 20. If the number itself is 4 times the sum of its digits, find the number.

21. Find the number of two digits in which the units' digit exceeds the tens' digit by 2, and such that twice the number exceeds the sum of the squares of the digits by 40.

22. The diagonal of a rectangle is 13. If the rectangle were 4 inches shorter and 2 inches wider, the diagonal would be $\sqrt{113}$. What are the dimensions of the rectangle?

23. Two boys on bicycles make a trip of 15 miles. The sum of their rates is 15 miles per hour. The first boy requires 50 minutes more for the trip than the second boy. What are their rates?

24. The area of a certain rectangular field is 260 rods. Its perimeter is 66 rods. What are its dimensions?

25. The area of a certain garden is 1000 square feet. If the length be decreased by 5 feet, and the width be increased by 5 feet, the area is increased by 50 square feet. What are the dimensions of the garden?

26. The area of a certain triangle is 108 square feet. Its base exceeds its altitude by 6 feet. What are the base and altitude of the triangle?

27. If the larger of certain two numbers be multiplied by the difference of the numbers, the result is 44. The product of the two numbers is 77. What are the numbers?

SUPPLEMENTARY EXAMPLES

211. Miscellaneous types and methods. Many systems of equations which cannot be solved by the methods already given may be solved by combining the equations so as to obtain a linear equation or an equation of the form xy = a constant.

Example 1. — Solve the system:

$$\begin{cases} x^2 + y^2 + 2x + 2y = 23 & (1) \\ xy = 6 & (2) \end{cases}$$

Solution. — 1. M_2 (2) $\quad 2\,xy = 12. \qquad (3)$

2. Adding (1) and (3), $x^2 + 2\,xy + y^2 + 2\,x + 2\,y = 35. \qquad (4)$

$$\therefore\ (x+y)^2 + 2(x+y) - 35 = 0.$$

$$\therefore\ (x+y+7)(x+y-5) = 0. \quad (\S\ 205)$$

$$\therefore\ x+y = -7, \text{ or } x+y = 5. \quad (\S\ 110)\ (5)$$

3. Form the systems: $A\begin{cases} x+y = -7 \\ xy = 6 \end{cases} \quad B\begin{cases} x+y = 5 \\ xy = 6 \end{cases}$

4. Solving A, $x = -1, y = -6$; or, $x = -6, y = -1$.
Solving B, $x = 3, y = 2$; or, $x = 2, y = 3$.

Check. — The four solutions check when substituted in equations (1) and (2).

Example 2. — Solve the system:

$$\begin{cases} m^2 + mn + n^2 = 7 & (1) \\ m + n = 5 + mn & (2) \end{cases}$$

Solution. — 1. Square (2),

$$m^2 + 2\,mn + n^2 = 25 + 10\,mn + m^2n^2. \qquad (3)$$

2. Subtract (1) from (3), $mn = 18 + 10\,mn + m^2n^2. \qquad (4)$

$$\therefore\ m^2n^2 + 9\,mn + 18 = 0. \qquad (5)$$

$$\therefore\ (mn+6)(mn+3) = 0; \quad \therefore\ mn = -6, \text{ or } mn = -3. \quad (\S\ 107)$$

3. Form the systems, $A\begin{cases} m+n = 5+mn \\ mn = -6 \end{cases} \quad B\begin{cases} m+n = 5+mn \\ mn = -3 \end{cases}$

4. Solving A, $m = 2, n = -3$; or, $m = -3, n = 2$.
Solving B, $m = 3, n = -1$; or, $m = -1, n = 3$.

Check. — The four solutions check when substituted in equations (1) and (2).

EXERCISE 159

1. $\begin{cases} a^2 + b^2 = 50 \\ ab + 7 = 0 \end{cases}$

2. $\begin{cases} x^2 - xy + y^2 = 7 \\ x + y = 4 \end{cases}$

3. $\begin{cases} 2\,a^2 - 3\,ab = 12 \\ ab = 20 \end{cases}$

4. $\begin{cases} 3\,x^2 + 8\,y^2 = 14 \\ x^2 - 12\,y^2 = 1 \end{cases}$

5. $\begin{cases} cd - 4\,d^2 = 1 \\ 3\,cd - c^2 = -10 \end{cases}$

6. $\begin{cases} 10\,x^2 + 11\,xy + y^2 = 324 \\ x + y = 3(y - x) \end{cases}$

7. $\begin{cases} x^2 + xy = 400 \\ x - y = 7 \end{cases}$

8. $\begin{cases} x^2 - y^2 = 9 \\ 5\,y = 3\,x \end{cases}$

9. $\begin{cases} x^2 + y^2 = 16 + 3x \\ x^2 - y^2 = 4 \end{cases}$

10. $\begin{cases} x^2 - xy + y^2 = 63 \\ x + y = 12 \end{cases}$

11. $\begin{cases} a^2 - b^2 = 16 \\ a^2 + 2b^2 = 64 \end{cases}$

12. $\begin{cases} 9x^2 - y^2 = 16 \\ y^2 - 14 = 3x \end{cases}$

13. $\begin{cases} a^2 - ab + b^2 = 21 \\ 2ab - b^2 = 15 \end{cases}$

14. $\begin{cases} cd + 4d^2 = 8 \\ c^2 + 3cd = 28 \end{cases}$

15. $\begin{cases} 2a^2 - ab + b^2 = 30 \\ a^2 = 2ab - b^2 \end{cases}$

16. $\begin{cases} p(1 + .03\,r) = 840 \\ p(1 + .07\,r) = 960 \end{cases}$

17. $\begin{cases} A^2B^2 - 3AB = 88 \\ A - B = 6 \end{cases}$

18. $\begin{cases} 2x^2y - x = -7 \\ 4x^4y^2 + x^2 = 37 \end{cases}$

19. $\begin{cases} x^2 + y^2 = 25 \\ (x + 1)^2 + (y - 1)^2 = 25 \end{cases}$

20. $\begin{cases} x^3 + 27y^3 = 756 \\ x^2 - 3xy + 9y^2 = 63 \end{cases}$

21. $\begin{cases} x(x + y) = 126 \\ y(x - y) = 20 \end{cases}$

22. $\begin{cases} 25m^2 + p^2 = 25 \\ 5mp = -12 \end{cases}$

23. $\begin{cases} p^2 - 4s^2 = 3 \\ ps = -1 \end{cases}$

24. $\begin{cases} x^2 + y^2 - (x + y) = 22 \\ x + y + xy + 1 = 0 \end{cases}$

25. $\begin{cases} x^2 + xy + y^2 = 61 \\ xy + x + y = 29 \end{cases}$

26. $\begin{cases} 8x^3 + y^3 = 7m^3 \\ 2x + y = m \end{cases}$

27. $\begin{cases} 2x^2 - 11y^2 = 8 \\ 3x^2 + 13y^2 = 248 \end{cases}$

28. $\begin{cases} x + y = 2a \\ x^2 + y^2 = 2a^2 + 2 \end{cases}$

29. $\begin{cases} x^2 + y^2 = 2(a^2 + b^2) \\ x - y = 2b \end{cases}$

30. $\begin{cases} x^2 + y^2 = 16a \\ x^2 + y^2 - 2x = 8a \end{cases}$

31. $\begin{cases} 7v^2 - 10vt - 12t^2 = 36 \\ v^2 + 6vt + 4t^2 = -4 \end{cases}$

32. $\begin{cases} x + y = b \\ xy = c \end{cases}$

33. $\begin{cases} x^2 - xy = ay \\ xy + ay^2 = b^2 \end{cases}$

34. $\begin{cases} (x + y)(a^2 - b^2) = (x - y)(a^2 + b^2) \\ xy = b^2 \end{cases}$

35. $\begin{cases} \dfrac{2m + n}{2m - n} + \dfrac{2m - n}{2m + n} = \dfrac{10}{3} \\ 4m^2 + n^2 = 45 \end{cases}$

XVIII. THE THEORY OF QUADRATIC EQUATIONS

212. The sum and the product of the roots of a quadratic equation bear interesting relations to the coefficients of the equation.

The general quadratic equation is

$$ax^2 + bx + c = 0. \tag{1}$$

Dividing both members by a,

$$x^2 + \frac{b}{a} \cdot x + \frac{c}{a} = 0. \tag{2}$$

In § 181, page 271, you proved that the roots of this equation are

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}; \text{ and } r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{3}$$

Using these two results:

I. $$\boxed{r_1 + r_2 = \frac{-2b}{2a} = -\frac{b}{a}} \tag{4}$$

II. $$r_1 r_2 = \frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{4a^2} = \frac{b^2 - (b^2 - 4ac)}{4a^2}.$$

$$\boxed{\therefore r_1 r_2 = \frac{4ac}{4a^2} = \frac{c}{a}} \tag{5}$$

These relations are expressed by the

Rules. — **In the quadratic equation $ax^2 + bx + c = 0$:**

1. The sum of the roots is $-\frac{b}{a}$. **From (4)**

2. The product of the roots is $\frac{c}{a}$. **From (5)**

3. **If the coefficient of x^2 is made 1 (as in equation (2) above), then the coefficient of x is minus the sum of the roots, and the constant term is the product of the roots.** (See (2), (4), (5).)

Example. — What are the sum and the product of the roots of ✓

$$2x^2 = 9x + 5?$$

Solution. — 1. $2x^2 = 9x + 5.$

2. $\therefore 2x^2 - 9x - 5 = 0.$

3. $\therefore a = 2; \quad b = -9; \quad c = -5.$

4. $\therefore r_1 + r_2 = -\frac{b}{a} = -\left(\frac{-9}{2}\right) = \frac{9}{2}.$

5. $\therefore r_1 r_2 = \frac{c}{a} = \frac{-5}{2} = -\frac{5}{2}.$

NOTE. — Always arrange the quadratic as in Step 2, with all terms on the left side, and make the coefficient of x^2 be positive.

EXERCISE 160

Find by this rule the sum and the product of the roots of the following equations. Check Examples 1, 2, 3, and 6 by finding the roots.

1. $x^2 + 8x + 7 = 0$

2. $z^2 - 9z + 14 = 0$

3. $8x^2 - 2x - 15 = 0$

4. $63y^2 - 16y + 1 = 0$

5. $4a = 7 - 3a^2$

6. $x^2 + 5bx = -4b^2$

7. $15x^2 - a^2 = 2ax$

8. $3x^2 = 4xt + 2t^2$

9. One root of $3x^2 - 2x - 5 = 0$ is -1. Find the other root.

Suggestion. — What is the sum of the roots?

10. One root of $2x^2 - 7x + 3 = 0$ is 3. Find the other root.

11. One root of $6x^2 + 1 = 5x$ is $-\frac{1}{6}$. Find the other root.

12. Find k so that one root of $x^2 + 4x - k = 0$ may be -12.

Solution. — 1. $r_1 + r_2 = -4. \quad \therefore -12 + r_2 = -4. \quad \therefore r_2 = 8.$

2. But $r_1 r_2 = -k. \quad \therefore -k = (-12) \cdot 8; \quad \therefore k = ?$

Check. — Substitute in $x^2 + 4x - k = 0$ the value of k you find; solve the equation, and thus determine whether -12 really is one root.

13. Find k so that $-\frac{2}{3}$ may be a root of $6x^2 + 7x + 2k = 0$.

14. Find k so that $\frac{3}{4}$ may be a root of $8x^2 - 10x = -k$.

15. Find a so that 7 may be a root of $7z^2 - 48z - a = 0$.

16. Find c so that the roots of $2z^2 - 8z + c = 0$ shall be equal.

Solution. — 1. Let the two equal roots be represented by r.

2. $\therefore\ r + r = ?\qquad \therefore\ r = ?$

3. But $r \cdot r = \frac{c}{2}. \qquad \therefore\ c = ?$

Check. — Substitute in $2z^2 - 8z + c = 0$ for c its value. Then solve the equation to determine whether the roots are really equal.

17. Find t so that the roots of $3x^2 - 4x + t = 0$ shall be equal.

18. Find s so that the roots of $4x^2 - sx + 9 = 0$ shall be equal.

19. Find m so that $x^2 - 14mx + 9 = 0$ shall have equal roots.

20. Find c so that $y^2 - 5y - 8c = 0$ shall have equal roots.

21. Find a so that the roots of $ax^2 - 3x + 4 = 0$ shall be equal.

213. Forming quadratic equations having given roots. There are two methods.

Example 1. — Form the equation whose roots are 2.4 and $-.5$.

Solution. — 1. Let the coefficient of x^2 be 1. Then the equation is $x^2 - (r_1 + r_2)x + r_1r_2 = 0$. (§ 212)

2. But $r_1 + r_2 = 2.4 + (-.5) = 1.9$; and $r_1 \cdot r_2 = 2.4 \cdot (-.5) = -1.2$.

3. $\therefore$ the equation is $x^2 - 1.9x - 1.2 = 0$.

Check. — Either solve the equation, obtaining its roots, or substitute the given roots in the equation.

Example 2. — Form the equation whose roots are $\frac{1}{2}$ and $-\frac{3}{4}$.

Solution. — 1. If $x = \frac{1}{2}$, then $x - \frac{1}{2} = 0$.

2. If $x = -\frac{3}{4}$, then $x + \frac{3}{4} = 0$.

3. $\therefore\ (x - \frac{1}{2})(x + \frac{3}{4}) = 0$, or $x^2 + \frac{1}{4}x - \frac{3}{8} = 0$.

4. $\therefore\ 8x^2 + 2x - 3 = 0$.

Check as in Example 1.

NOTE. — This second method may be used when forming an equation which shall have three or more given roots.

EXERCISE 161

Form the equation whose roots shall be

1. 2, 5
2. $-3, 1$
3. $6, -1$
4. $-3, -7$
5. $\frac{1}{2}, \frac{3}{4}$
6. $\frac{2}{3}, -\frac{1}{6}$
7. 1.5, 2
8. .2, .5
9. $1.5, -2.4$
10. $-3.6, -.25$
11. $+3m, +2m$
12. $-4t, +7t$
13. $\frac{1}{2}c, -\frac{2}{3}c$
14. $d + e, d - e$
15. $\sqrt{5}, -\sqrt{5}$
16. $1 + \sqrt{2}, 1 - \sqrt{2}$
17. $\sqrt{3} + 2, \sqrt{3} - 2$
18. $3 + \sqrt{6}, 3 - \sqrt{6}$
19. $b + 2c, b - 2c$
20. $\sqrt{m}, -\sqrt{m}$
21. $a + \sqrt{b}, a - \sqrt{b}$

DETERMINATION OF THE CHARACTER OF THE ROOTS

214. Classification of numbers. The numbers considered in this text to this point are:

I. The Real Numbers.
  1. Rational numbers.
    a. The positive and negative integers.
    b. The positive and negative fractions whose terms are integers.
  2. Irrational numbers.
    a. Quadratic surds. (§ 170)
    b. Surd expressions, such as $2 + \sqrt{3}$. (§ 171)

II. Imaginary Numbers.
  1. Pure imaginaries, such as $\sqrt{-3}$. (§ 189)
  2. Complex numbers, such as $2 - \sqrt{-3}$.

215. Sometimes it is desired to know what kind of roots a quadratic has, without any reference to the exact values of these roots.

Thus, the roots of $2x^2 - 8x + 3 = 0$ are $\frac{8 \pm \sqrt{64 - 24}}{4}$.

Since $64 - 24 = 40$, the roots are real numbers.

They would be imaginary only if the number under the radical sign were negative.

Since 40 is not a perfect square, the roots are irrational. (See § 170.)

Since 40 is added to 8 to get one root, and subtracted from 8 to get the other root, the roots are unequal.

Hence the roots are real, irrational, and unequal.

It is possible, however, to determine the nature of the roots without first finding the roots, as in the example just given.

The roots of $ax^2 + bx + c = 0$ are, as you know,

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \text{ and } r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Rule. — 1. If $b^2 - 4ac$ is positive, the roots are real and unequal. They are rational if $b^2 - 4ac$ is a perfect square.

2. If $b^2 - 4ac$ = zero, the roots are real, rational, and equal. (Why?)

3. If $b^2 - 4ac$ is negative, the roots are imaginary.

NOTE. — $b^2 - 4ac$ is called the **discriminant** of the quadratic. (Why?)

Example 1. — What kind of roots has

$$2x^2 - 5x - 18 = 0?$$

Solution. — 1. $b^2 - 4ac = (-5)^2 - 4(2)(-18)$

2. $= 25 + 144 = 169 = 13^2$.

3. $\therefore$ by Rule 1, the roots are real, rational, and unequal.

Example 2. — What kind of roots has

$$3x^2 + 2x + 1 = 0?$$

Solution. — 1. $b^2 - 4ac = 4 - 4 \cdot 3 \cdot 1$

2. $= 4 - 12$, or -8.

3. $\therefore$ by Rule 3, the roots are imaginary.

Example 3. — What kind of roots has

$$4x^2 - 12x + 9 = 0?$$

Solution. — 1. $b^2 - 4ac = 144 - 4 \cdot 4 \cdot 9 = 144 - 144 = 0.$
2. $\therefore$ by Rule 2, the roots are real, rational, and equal.

NOTE. — This type of quadratic is most easily understood if the quadratic is solved by factoring. This example becomes $(2x - 3)(2x - 3) = 0$. The roots are $\frac{3}{2}$ and $\frac{3}{2}$.

EXERCISE 162

Determine by inspection the kind of roots of:

1. $y^2 - 7y + 12 = 0$
2. $9x^2 - 24x + 16 = 0$
3. $2m^2 - m + 1 = 0$
4. $3y^2 = 5y + 2$
5. $7x^2 + 5x = 9$
6. $1 + 6t + 4t^2 = 0$
7. $3x - 9x^2 + 1 = 0$
8. $x^2 = 2x - 7$
9. $10 = 4x + 5x^2$
10. $16m^2 - 4 = 0$
11. $9x^2 - 30x + 25 = 0$
12. $x^2 - 4x + 8 = 0$

XIX. LITERAL, FRACTIONAL, AND NEGATIVE EXPONENTS

216. In the preceding chapters, only positive integers have been used as exponents. The fundamental definition when m is a positive integer is

$$a^m = a \cdot a \cdot a \ldots. a \qquad (m \text{ factors})$$

217. There are five fundamental laws of exponents. The proofs follow, for the case when m and n are positive integers.

I. Law of exponents for multiplication.

Just as $a^5 \times a^7 = a^{12}$, so $\boldsymbol{a^m \times a^n = a^{m+n}}$.

Proof. — 1. $a^m = a \cdot a \cdot a \ldots.. a$ $(m$ factors)

2. $a^n = a \cdot a \cdot a \ldots.. a$ $(n$ factors)

3. $\therefore\ a^m \cdot a^n = \underset{(m \text{ factors})}{\{a \cdot a \ldots.. a\}}\ \underset{(n \text{ factors})}{\{a \cdot a \ldots.. a\}}$

4. $= a \cdot a \ldots.. a \cdot a \cdot a \ldots.. a$ $(m + n)$ factors

5. $\therefore\ a^m \cdot a^n = a^{m+n}.$ § 216

II. Law of exponents for division.

Just as $a^9 \div a^4 = a^5$, so $\boldsymbol{a^m \div a^n = a^{m-n}}$ (m greater than n).

Proof. — 1. $\dfrac{a^m}{a^n} = \dfrac{\cancel{a} \cdot \cancel{a} \cdot \cancel{a} \ldots. \cancel{a} \cdot a \cdot a \ldots.. a \quad (m \text{ factors})}{\cancel{a} \cdot \cancel{a} \cdot \cancel{a} \ldots. \cancel{a} \quad (n \text{ factors})}$

2. $= a \cdot a \ldots.. a$ $(m - n)$ factors

3. $\therefore\ a^m \div a^n = a^{m-n}.$ § 216

III. Law of exponents for finding a power of a power.

Just as $(a^5)^3 = a^{15}$, so $\boldsymbol{(a^m)^n = a^{mn}}$.

Proof. — 1. $(a^m)^n = a^m \cdot a^m \cdot a^m \ldots.. a^m$ $(n$ factors)

2. $= a^{m+m+m \cdots\cdots m}$ $(n$ terms)

3. $\therefore\ (a^m)^n = a^{mn}.$

IV. Law of exponents for finding a power of a product.

Just as $(ab)^5 = a^5b^5$, so $(\boldsymbol{ab})^n = \boldsymbol{a^nb^n}$.

Proof. — 1. $(ab)^n = (ab) \cdot (ab) \cdot (ab) \ldots\ldots (ab)$ (n factors)

2. $\therefore (ab)^n = \underbrace{(a \cdot a \cdot a \ldots\ldots a)}_{(n \text{ factors})}\underbrace{(b \cdot b \cdot b \ldots\ldots b)}_{(n \text{ factors})}$

3. $\therefore (ab)^n = a^n \cdot b^n.$

V. Law of exponents for finding a power of a quotient.

Just as $\left(\frac{a}{b}\right)^5 = \frac{a^5}{b^5}$, so $\left(\frac{\boldsymbol{a}}{\boldsymbol{b}}\right)^n = \frac{\boldsymbol{a^n}}{\boldsymbol{b^n}}$.

Proof. — 1. $\left(\frac{a}{b}\right)^n = \left(\frac{a}{b}\right) \cdot \left(\frac{a}{b}\right) \cdot \left(\frac{a}{b}\right) \ldots\ldots \left(\frac{a}{b}\right)$ (n factors)

2. $\therefore \left(\frac{a}{b}\right)^n = \frac{a \cdot a \cdot a \ldots\ldots a}{b \cdot b \cdot b \ldots\ldots b}$ (n factors) (n factors)

3. $\therefore \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$

Involution is the name given to the process of finding a power of a number.

EXERCISE 163

In the following examples, *the literal exponents represent positive integers.* Find the results of the indicated operations, using the five laws above.

1. $x^{10} \cdot x^4$
2. $m^6 \cdot m^8$
3. $y^4 \cdot y^a$
4. $x^a \cdot x^b$
5. $x^{2n} \cdot x^{5n}$
6. $x^3 \cdot x^r$
7. $y^{c-2} \cdot y^3$
8. $z^{2n} \cdot z^5$
9. $w^{n-1} \cdot w^n$
10. $x^{3m} \cdot x^{m+1}$
11. $y^{10} \div y^3$
12. $z^{12} \div z^n$
13. $t^{2n} \div t^3$
14. $x^{3m} \div x^m$
15. $y^{m+4} \div y^3$
16. $z^m \div z^{m-1}$
17. $k^{r+2} \div k^r$
18. $w^{3r+1} \div w^2$
19. $x^{n+3} \div x^2$
20. $x^{n-3} \div x^2$
21. $(x^3)^2$
22. $(y^2)^3$
23. $(z^4)^3$
24. $(w^6)^3$
25. $(x^5)^6$
26. $(y^3)^7$
27. $(m^4)^9$
28. $(-a^2b)^2$
29. $(-2ab)^3$
30. $(2xy)^3$
31. $(3xy^2z)^3$
32. $(4mn^2)^3$
33. $(-2x^2y)^3$
34. $(-4a^3b)^3$
35. $(-\frac{1}{2}a)^3$
36. $(-\frac{2}{3}cd^2)^3$
37. $(\frac{3}{4}xy^2)^3$
38. $(x^2y^2w)^4$
39. $(-a^2b^3)^5$
40. $(a^3)^n$
41. $(b^m)^4$
42. $(x^2y^3)^n$
43. $(r^4s^3)^n$
44. $(x^my^n)^t$

45. $\left(\frac{x^2}{y^3}\right)^2$

46. $\left(\frac{2a}{b}\right)^3$

47. $\left(\frac{3c^2}{5d}\right)^3$

48. $\left(-\frac{2m^3}{3n^2}\right)^3$

49. $\left(\frac{x^2}{y}\right)^4$

50. $\left(\frac{m^3}{n^2}\right)^5$

51. $\left(\frac{a^2}{b^3}\right)^5$

52. $\left(-\frac{c}{d^2}\right)^7$

53. $\left(\frac{c^2}{d^3}\right)^6$

54. $\left(\frac{r^2}{s}\right)^n$

55. $\left(\frac{x^3}{y}\right)^m$

56. $\left(\frac{x^m}{y^n}\right)^2$

57. $\left(\frac{x^n}{y^2}\right)^3$

58. $\left(\frac{x^{2a}}{y^b}\right)^3$

59. $\left(\frac{a^n}{b^m}\right)^r$

60. $\left(\frac{x^{2p}}{y^{3p}}\right)^n$

218. Cube and other roots appear in pure and applied mathematics.

Just as a number has *two square roots,* so it has *three cube roots, four fourth roots,* and *n nth roots.*

Thus, consider the fourth roots of 16.

1. $+2$ is one, since $(+2)^4 = 16$.
2. -2 is one, since $(-2)^4 = 16$.
3. $+2i$ is one, since $(+2i)^4 = 16i^4 = 16 \times 1 = 16$. (§ 189)
4. $-2i$ is one, since $(-2i)^4 = 16i^4 = 16 \times 1 = 16$.

Of these four roots, $+2$ is called the *principal fourth root* of 16.

The principal even root of a positive number is a positive number.

One of the three cube roots of $+8$ is $+2$, since $(+2)^3 = 8$.

The other two are *complex numbers.* $+2$ is called the *principal cube root* of $+8$.

One of the three cube roots of -8 is -2, since $(-2)^3 = -8$.

The other two are complex numbers. -2 is called the *principal cube root* of -8.

The principal odd root of a positive number is a positive number; the principal odd root of a negative number is a negative number.

The principal nth root of a number x is indicated by the symbol $\sqrt[n]{x}$. The number n is called the index of the root. The number under the radical sign is called the **radicand.**

We know that $\sqrt{x^6} = x^3$, since $(x^3)^2 = x^6$;

that $\sqrt[3]{-y^6} = -y^2$, since $(-y^2)^3 = -y^6$;

that $\sqrt[4]{81\,x^{12}y^8} = \sqrt[4]{3^4x^{12}y^8} = 3\,x^3y^2$, since $(3\,x^3y^2)^4 = 81\,x^{12}y^8$;

that $\sqrt[5]{-32\,x^5y^{10}} = \sqrt[5]{(-2)^5x^5y^{10}} = -2\,xy^2$,

since $(-2\,xy^2)^5 = -32\,x^5y^{10}$.

Rule. — To find the principal root of a monomial:

1. Write each factor of the monomial with an exponent which is its original exponent divided by the index of the root.

2. If the radicand is positive, the root is real and positive, whether the index is even or odd.

3. If the radicand is negative, the root is:

a. real and negative, if the index is odd;

b. imaginary, if the index is even.

(In this chapter, no imaginary roots occur.)

Historical Note. — A symbol for extracting a root did not appear until the fifteenth century. In Italian mathematics, the first letter of the word Radix, meaning the root, was used to indicate the square root: thus, ℞. Presently there were used ℞.2^a, ℞.3^a, etc., to indicate the square, cube, and other roots. Chuquet, a French mathematician of about 1500, used ℞^{2}, ℞^{3}, etc.

In Germany, a point was placed before a number to indicate that its square root was to be taken. Two points were used to indicate the fourth root, and three the third root. Reise, 1492–1559, replaced the point by the symbol $\surd$, to indicate the square root, and Rudolph, 1515, used the symbol, $\surd\surd$, for the fourth root. Stevin, 1548–1620, used the better symbols: $\surd$②, $\surd$③, etc. Girard, 1590–1632, used $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, etc. Descartes used the vinculum to indicate what numbers were affected by the root.

EXERCISE 164

Give the principal root indicated below:

1. $\sqrt[3]{a^6b^3}$
2. $\sqrt[3]{27\,x^3y^6}$
3. $\sqrt[3]{-8\,a^6}$
4. $\sqrt[3]{-125\,r^{12}}$
5. $\sqrt[3]{-27\,m^3y^6}$
6. $\sqrt[3]{64\,r^6y^6}$
7. $\sqrt[3]{\frac{1}{8}\,a^3}$
8. $\sqrt[3]{-\frac{1}{27}\,r^3s^3}$
9. $\sqrt[4]{x^{12}}$
10. $\sqrt[4]{16\,a^4b^4}$
11. $\sqrt[4]{81\,x^4y^8}$
12. $\sqrt[4]{256\,m^4}$

13. $\sqrt[4]{\frac{1}{16}x^4y^{12}}$

14. $\sqrt[5]{32a^{10}}$

15. $\sqrt[5]{-243a^5b^5}$

16. $\sqrt[5]{-32x^5z^{15}}$

17. $\sqrt[5]{-\frac{1}{32}r^5s^5}$

18. $\sqrt[6]{x^{12}y^6}$

19. $\sqrt[6]{64a^6b^{12}}$

20. $\sqrt[3]{8x^{3n}}$

21. $\sqrt[3]{-\frac{1}{27}x^{3n}y^{3n}}$

22. $\sqrt[4]{x^{4m}y^{8n}}$

23. $\sqrt[5]{-x^{5n}y^{10}}$

24. $\sqrt[6]{x^{12r}y^{6r}}$

25. $\sqrt[3]{\frac{x^6}{y^6}}$

26. $\sqrt[3]{-\frac{8a^6}{b^{12}}}$

27. $\sqrt[3]{-\frac{64m^6}{27n^3}}$

28. $\sqrt[3]{\frac{x^{3n}}{y^6}}$

29. $\sqrt[3]{\frac{125}{z^{12}}}$

30. $\sqrt[4]{\frac{x^8}{y^{12}}}$

31. $\sqrt[5]{-\frac{m^{10}}{n^{15}}}$

32. $\sqrt[6]{\frac{x^{12}}{y^6}}$

33. $\sqrt[4]{\frac{16a^4}{b^8}}$

34. $\sqrt[5]{\frac{-32m^5}{n^{10}}}$

35. $\sqrt{\frac{x^{2p}}{y^{6p}}}$

36. $\sqrt[3]{\frac{a^{3n}}{b^{3m}}}$

37. $\sqrt[5]{\frac{-x^{5r}}{y^{10s}}}$

38. $\sqrt[4]{\frac{x^{4a}}{y^{4b}}}$

39. $\sqrt[n]{\frac{x^{mn}}{y^{rn}}}$

219. Negative numbers, fractions, and zero also are used as exponents. So far, however, such expressions as a^{-3}, $a^{\frac{5}{3}}$, and a^0 have not been given any meaning in this text.

Observe that a^{-3} cannot mean that a is used as a factor -3 times, since doing anything -3 times does not have meaning.

220. Meaning of a fractional exponent.

If $a^{\frac{5}{3}}$ is to obey the law of multiplication on page 332, then

$$a^{\frac{5}{3}} \cdot a^{\frac{5}{3}} \cdot a^{\frac{5}{3}} = a^{\frac{15}{3}} = a^5.$$

$$\text{That is} \quad (a^{\frac{5}{3}})^3 = a^5, \text{ or } a^{\frac{5}{3}} = \sqrt[3]{a^5}.$$

This suggests the

Definition. — In a fractional exponent, the denominator indicates a root of the base, and the numerator a power of the base.

Thus, $a^{\frac{3}{4}} = \sqrt[4]{a^3}$; also $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$.

EXERCISE 165

Express with radical signs and find the value of:

1. $8^{\frac{1}{3}}$
2. $16^{\frac{1}{4}}$
3. $32^{\frac{1}{5}}$
4. $64^{\frac{1}{3}}$
5. $64^{\frac{1}{6}}$
6. $100^{\frac{1}{2}}$
7. $81^{\frac{1}{4}}$
8. $256^{\frac{1}{2}}$
9. $125^{\frac{1}{3}}$
10. $(-1000)^{\frac{1}{3}}$
11. $(-32a^5)^{\frac{1}{5}}$
12. $(-x^6y^3)^{\frac{1}{3}}$
13. $(-64a^3b^6)^{\frac{1}{3}}$
14. $(81m^4)^{\frac{1}{4}}$
15. $(-z^{10}w^5)^{\frac{1}{5}}$
16. $(m^{12})^{\frac{1}{6}}$
17. $(\frac{1}{64}a^6)^{\frac{1}{6}}$
18. $(-27m^3)^{\frac{1}{3}}$
19. $(-243a^5)^{\frac{1}{5}}$
20. $(-125x^6)^{\frac{1}{3}}$
21. $(\frac{1}{36}m^4)^{\frac{1}{2}}$
22. $(-\frac{1}{27}a^3)^{\frac{1}{3}}$
23. $(\frac{1}{121}m^6)^{\frac{1}{2}}$
24. $(-\frac{1}{32}x^{10})^{\frac{1}{5}}$

Express with radical signs:

25. $3^{\frac{2}{3}}$
26. $4^{\frac{3}{4}}$
27. $5^{\frac{3}{5}}$
28. $(2x)^{\frac{2}{3}}$
29. $(-3x)^{\frac{2}{3}}$
30. $(3m)^{\frac{3}{2}}$
31. $(-3m)^{\frac{3}{2}}$
32. $(5a)^{\frac{1}{2}}$
33. $5 \cdot a^{\frac{1}{2}}$
34. $2ab^{\frac{2}{3}}$
35. $(2ab)^{\frac{2}{3}}$
36. $x^{\frac{3}{4}}y^{\frac{4}{3}}$
37. $x^{\frac{5}{8}}$
38. $(-y)^{\frac{4}{5}}$
39. $-y^{\frac{4}{5}}$

Express with fractional exponents:

40. $\sqrt[4]{m^3}$
41. $\sqrt[5]{x^2}$
42. $\sqrt[3]{3a^2}$
43. $3\sqrt[3]{a^2}$
44. $\sqrt[5]{m^1}$
45. $\sqrt[6]{a^4b^3}$
46. $5\sqrt[3]{a^2}$
47. $3\sqrt[4]{y^2}$
48. $2a\sqrt[3]{m^2}$
49. $3b^2\sqrt[5]{a}$

221. Meaning of a zero exponent.

If a^0 is to obey the multiplication law, then

$$a^m \cdot a^0 = a^{m+0} = a^m.$$

$$\therefore \quad a^0 = a^m \div a^m = 1.$$

This suggests the

Definition. — The zero power of any number, except zero, is 1.

Thus, $5^0 = 1$; $(5000)^0 = 1$; $x^0 = 1$; $(-15)^0 = 1$.

222. Meaning of a negative exponent.

If a^{-3} is to obey the multiplication law, then

$$a^{-3} \cdot a^{+3} = a^{-3+3} = a^0 = 1.$$

This suggests the

Definition. — $$a^{-n} = \frac{1}{a^n}.$$

Thus, $x^{-4} = \dfrac{1}{x^4}$; $y^{-\frac{2}{3}} = \dfrac{1}{y^{\frac{2}{3}}} = \dfrac{1}{\sqrt[3]{y^2}}$; $(2)^{-3} = \dfrac{1}{(2)^3} = \dfrac{1}{8}$.

EXERCISE 166

Express with positive exponents and find the value of:

1. 4^{-2}
2. 3^{-3}
3. 2^{-5}
4. 5^{-2}
5. 8^0
6. 6^{-2}
7. $(-3)^{-2}$
8. $(-2)^{-3}$
9. $(-5)^{-2}$
10. $(-2)^{-4}$
11. 6^{-3}
12. $8^{-\frac{1}{3}}$
13. $16^{-\frac{1}{2}}$
14. $81^{-\frac{1}{4}}$
15. $2^{-1} \cdot 5^{-2}$
16. $6^0 \cdot 3^{-3}$
17. $64 \cdot 2^{-5}$
18. $9^0 \cdot 9^{-\frac{1}{2}}$
19. $(-8)^{-\frac{1}{3}}$
20. $(-64)^{-\frac{1}{3}}$
21. $(-32)^{-\frac{1}{5}}$
22. $4^{-\frac{3}{2}}$
23. $81^{-\frac{1}{4}} \cdot 6$
24. $(-27)^{-\frac{1}{3}} \cdot 9^{\frac{1}{2}}$
25. $16^{-\frac{1}{2}} \cdot 64^{\frac{1}{3}}$
26. $18^0 \cdot 2^{-6}$
27. $3^{-2} \cdot 27$
28. $4^{-3} \cdot 256^{\frac{1}{4}}$

Rewrite with only positive exponents:

29. a^3b^{-4}
30. $(3x)^{-3}$
31. $2r^{-4}$
32. $(2r)^{-4}$
33. $2a^{-2}b^4$
34. $(2a)^{-2}b^4$
35. $10^{-2} \cdot x^5$
36. $10^{-3}a^2b^3$

223. Negative exponents in fractions.

Example 1. — $\dfrac{3a^{-3}}{4c^{-2}} = 3 \cdot \dfrac{1}{a^3} \div 4 \cdot \dfrac{1}{c^2} = \dfrac{3}{a^3} \div \dfrac{4}{c^2} = \dfrac{3}{a^3} \cdot \dfrac{c^2}{4} = \dfrac{3c^2}{4a^3}.$

Observe that a^{-3} of the numerator becomes a^{+3} in the denominator; that c^{-2} of the denominator becomes c^{+2} in the numerator.

This suggests the

Rule. — Any factor of one term of a fraction may be transferred to the other term provided the sign of its exponent is changed.

Example 2. — $\dfrac{5\,x^2y^{-4}z^3}{w^{-2}t} = \dfrac{5\,x^2w^2z^3}{y^4t}.$

Example 3. — $\dfrac{3\,a^2b}{cd^3} = 3\,a^2bc^{-1}d^{-3}.$

EXERCISE 167

Rewrite with only positive exponents:

1. $\dfrac{x^{-3}y}{z^2}$

2. $\dfrac{3\,m}{n^{-2}}$

3. $\dfrac{4\,a^{-2}}{b^3}$

4. $\dfrac{5\,x^{-3}}{y^{-2}}$

5. $\dfrac{mn}{10^{-5}}$

6. $\dfrac{x^2}{(2\,y)^{-2}}$

7. $\dfrac{5\,m^2n^{-4}}{6\,r^2}$

8. $\dfrac{4\,a^{-3}b^2}{5\,c^2d^{-3}}$

9. $\dfrac{8\,a^{\frac{1}{2}}b^{-\frac{1}{3}}}{c^{\frac{1}{3}}}$

10. $\dfrac{k}{10^{-15}}$

11. $\dfrac{10^{-7}x^2}{y^3}$

12. $\dfrac{4 \cdot 3^{-1}\pi}{r^{-3}}$

Write without any denominator:

13. $\dfrac{5\,x^6}{y^3}$

14. $\dfrac{3\,c^4}{d^{-2}}$

15. $\dfrac{2\,m^3n^2}{r^5}$

16. $\dfrac{x^3y^{-2}}{z^4}$

17. $\dfrac{7\,cd}{m^4}$

18. $\dfrac{8\,x^{\frac{1}{2}}}{y^{\frac{2}{3}}}$

19. $\dfrac{5\,a^3b^{-2}}{2^{-1}c^{-2}}$

20. $\dfrac{6\,a^{\frac{2}{3}}}{b^{-\frac{3}{4}}}$

224. **It can be proved that the five fundamental laws of page 332 apply for any rational exponent.** The laws will be assumed in this text.

EXERCISE 168

I. The multiplication law. (Page 332.)

Example. — $a^5 \cdot a^{-2} \cdot a^0 \cdot a^{\frac{1}{2}} = a^{5-2+0+\frac{1}{2}} = a^{3\frac{1}{2}}.$

1. Express in words the multiplication law.

2. Multiply each of the following numbers:

$$r^3; \quad s^{-3}; \quad r^2s^3; \quad r^{-3}s^{-4}; \quad r^ms^n$$

a. by r^4 *b.* by s^{-3} *c.* by r^3s^2 *d.* by $r^{-2}s^{-2}$

3. Multiply each of the following numbers:

$$x^{\frac{1}{3}}; \quad y^{\frac{3}{4}}; \quad x^{\frac{1}{2}}y^{\frac{2}{3}}; \quad x^ay^b$$

a. by $x^{\frac{1}{2}}$ *b.* by $y^{\frac{1}{4}}$ *c.* by $x^{\frac{1}{3}}y^{\frac{2}{3}}$ *d.* by $x^{\frac{1}{4}}y^{\frac{3}{4}}$

4. Multiply each of the following numbers:

$$m^{-\frac{1}{2}}; \quad n^{-\frac{1}{3}}; \quad m^{-\frac{1}{4}}n^{-\frac{2}{3}}; \quad m^{-\frac{3}{4}}n^{-\frac{1}{2}}$$

a. by m^2 *b.* by $m^{\frac{1}{2}}$ *c.* by $n^{-\frac{1}{4}}$ *d.* by $m^{-1}n^2$

Multiply:

5. $a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$ by $a^{\frac{1}{3}} + b^{\frac{1}{3}}$

6. $2x^{-2} - 3x^{-1} - 4$ by $3x^{-1} + 5$

7. $x^{\frac{3}{4}} + x^{\frac{1}{2}}y^{\frac{1}{4}} + x^{\frac{1}{4}}y^{\frac{1}{2}} + y^{\frac{3}{4}}$ by $x^{\frac{1}{4}} - y^{\frac{1}{4}}$

8. $x^{-2} + x^{-1}y^{-1} + y^{-2}$ by $x^{-2} - x^{-1}y^{-1} + y^{-2}$

9. $(x^{\frac{1}{3}} + y^{\frac{1}{3}})(x^{\frac{1}{3}} - y^{\frac{1}{3}})$

10. $(x^{-2} + y^{-2})(x^{-2} - y^{-2})$

11. $(a^{-3} + 4)(a^{-3} + 4)$

12. $(x^m - y^n)^2$

13. $(x^{-3} + 6y^{-1})\, x^{-3} + 2y^{-1})$

14. $(a^{\frac{1}{2}} - 2b^{\frac{1}{4}})(a^{\frac{1}{2}} + 3b^{\frac{1}{4}})$

15. $(x + x^{-1})^2$

16. $(x^a + y^b)(x^a - y^b)$

17. $(x^a - 3b)(x^a + 5b)$

18. $(x^m + 4y^n)(x^m - 7y^n)$

19. $(a^{\frac{3}{5}} - 7)(a^{\frac{3}{5}} + 9)$

20. $(c^{-\frac{1}{2}} + d^{-\frac{1}{3}})(c^{-\frac{1}{2}} - d^{-\frac{1}{3}})$

II. The division law. (Page 332.)

Example. — $m^{3\frac{1}{4}} \div m^{-\frac{1}{4}} = m^{3\frac{1}{4}-(-\frac{1}{4})} = m^{3\frac{1}{2}}.$

21. Express Law II in words.

22. Divide each of the following numbers:

$$t^7; \quad t^{-8}; \quad t^m; \quad t^{\frac{1}{3}}; \quad t^{-\frac{1}{4}}$$

a. by t^3 *b.* by t^{-2} *c.* by $t^{\frac{1}{2}}$ *d.* by $t^{-\frac{1}{3}}$

23. Divide each of the following numbers:

$$x^{-2}y^3;\quad x^my^n;\quad x^{\frac{2}{3}}y^{\frac{1}{3}};\quad x^{-\frac{3}{4}}y^{\frac{1}{2}}$$

a. by x^2y^3 *b.* by $x^{-2}y^{-1}$ *c.* by $x^{\frac{1}{4}}y^{\frac{1}{3}}$

Divide:

24. $a^2 + 2a + 1$ by a^4

25. $6x^{-2} - 8x^{-3} + 4x^{-4}$ by $2x^{-1}$

26. $m^2 - m + 1$ by $m^{\frac{1}{2}}$ **27.** $3c^2 - 6cd - 9d^2$ by $3cd$

28. $4r^{\frac{3}{2}} - 6rs - 10s^{\frac{3}{2}}$ by $2r^{\frac{1}{2}}s^{\frac{1}{2}}$

29. $a - b$ by $a^{\frac{1}{2}} + b^{\frac{1}{2}}$ **31.** $a^2 - b^2$ by $a^{\frac{1}{2}} - b^{\frac{1}{2}}$

30. $a - b$ by $a^{\frac{1}{3}} - b^{\frac{1}{3}}$ **32.** $x^2 + y^2$ by $x^{\frac{2}{3}} + y^{\frac{2}{3}}$

33. $c^3 - d^3$ by $c^{\frac{3}{4}} - d^{\frac{3}{4}}$

III. Power of a power law. (Page 332.)

Example. — $(x^{-\frac{3}{2}})^{\frac{4}{3}} = x^{-\frac{3}{2}\cdot\frac{4}{3}} = x^{-2}.$

34. What is the value of each of the following numbers?

$$(x^3)^n;\quad (y^{-4})^n;\quad (z^{\frac{2}{3}})^n;\quad (w^{-\frac{3}{4}})^n$$

when n is: *a.* 2 *b.* -3 *c.* $\frac{1}{3}$ *d.* $-\frac{1}{2}$ *e.* $-\frac{2}{3}$

35. $(10^2)^3 = ?$ **37.** $(10^{2.6})^4 = ?$ **39.** $(10^{2.56})^{\frac{1}{2}} = ?$

36. $(10^{1.5})^2 = ?$ **38.** $(10^{-1.3})^2 = ?$ **40.** $(10^{-9.824})^{\frac{1}{3}} = ?$

IV and V. Power of a product and power of a quotient. (Page 332.)

Example 1. — $(x^{-3}y^{\frac{1}{2}})^4 = x^{-12}y^2.$

Example 2. — $\left(\dfrac{a^{-\frac{2}{3}}}{b^{\frac{1}{2}}}\right)^{-6} = \dfrac{a^{-6\,\cdot\,(-\frac{2}{3})}}{b^{-6\,\cdot\,(\frac{1}{2})}} = \dfrac{a^4}{b^{-3}} = a^4b^3.$

41. Express Law IV in words.

42. Express Law V in words.

43. Give the simplified value of each of the following numbers:

$$(a^2b^{-3})^n;\quad (m^{-2}p^{\frac{1}{3}})^n;\quad (x^{-\frac{2}{3}}y^{\frac{3}{4}})^n;\quad (r^as^{-b})^n$$

when n is: *a.* 3 *b.* -2 *c.* $\frac{1}{3}$ *d.* $\frac{1}{2}$

Find the vaue of:

44. $(-125)^{\frac{2}{3}}$

Solution. — $(-125)^{\frac{2}{3}} = ((-5)^3)^{\frac{2}{3}} = (-5)^2 = 25.$

45. $9^{\frac{3}{2}}$

46. $8^{\frac{4}{3}}$

47. $16^{\frac{3}{4}}$

48. $25^{\frac{5}{2}}$

49. $27^{\frac{2}{3}}$

50. $64^{\frac{4}{3}}$

51. $(-125)^{\frac{4}{3}}$

52. $81^{\frac{3}{4}}$

53. $(-8)^{\frac{5}{3}}$

54. $(-27)^{\frac{4}{3}}$

55. $(9x^2)^{\frac{3}{2}}$

56. $(-27a^3)^{\frac{2}{3}}$

57. $(-32x^5)^{\frac{3}{5}}$

58. $(16x^8)^{\frac{3}{4}}$

59. $(36a^4)^{\frac{3}{2}}$

60. $(-1000c^3d^6)^{\frac{2}{3}}$

61. $(-8)^{-\frac{2}{3}}$

62. $(16)^{-\frac{1}{4}}$

63. $(25)^{-\frac{3}{2}}$

64. $(-32)^{-\frac{4}{5}}$

65. What is the meaning of:

a. $10^{.5}$ *b.* $10^{.25}$ *c.* $10^{1.5}$ *d.* $10^{1.75}$

66. Simplify: *a.* $\dfrac{10^{2.1} \times 10^{1.4}}{10^{1.2}}$ *b.* $\dfrac{10^{1.25} \times 10^{3.40}}{10^{2.6}}$

67. Multiply each of the following numbers:

$10^{1.25}$; $10^{2.1}$; $10^{9.73}$

a. by 10 *b.* by 100 *c.* by $10^{1.3}$

68. Divide each of the numbers in Example 67:

a. by 10 *b.* by $10^{1.05}$ *c.* by 100

69. Find: *a.* $(10^{1.6243})^{\frac{1}{2}}$ *b.* $(10^{3.4275})^{\frac{1}{3}}$

70. Find: *a.* $(10^{.3010})^2$ *b.* $(10^{2.4771})^3$

XX. RADICALS

225. A **radical** is the principal root of a number, indicated by a radical sign: as $\sqrt{5}$, $\sqrt[3]{a}$, or $\sqrt[4]{x+1}$.

If the indicated root can be obtained, the result is a **rational number**; if it cannot be obtained, it is an **irrational number.**

226. The **order** of a radical is determined by the **index** of the radical. (Page 334.)

Thus, a cube root is a radical of the third order;
a fourth root is a radical of the fourth order.

227. Radicals of the second order are most important and will be emphasized. Give the approximate decimal value of final results for radicals of second order. In many lists, in this text, the examples involving radicals of higher order are given separately and may be omitted unless the teacher has special reasons for teaching them.

228. Two fundamental principles of radicals:

I. $(\sqrt[n]{x})^n = x$.

Thus, $(\sqrt[3]{19})^3 = 19$; $(\sqrt[5]{3})^5 = 3$.

Really this is only the definition of n^{th} root; that is, the nth root of x is the number whose n^{th} power is x.

II. $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$. Thus, $\sqrt[4]{7 \cdot 9} = \sqrt[4]{7} \cdot \sqrt[4]{9}$.

Proof. — $\sqrt[n]{ab} = (ab)^{\frac{1}{n}} = a^{\frac{1}{n}} \cdot b^{\frac{1}{n}} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.

Expressed in words: *the nth root of the product of two or more numbers equals the product of the nth roots of the numbers.*

REDUCTION OF A RADICAL TO SIMPLEST FORM

229. Reducing a radical to an equivalent radical of lower order.

Example 1. — $\sqrt[6]{125} = \sqrt[6]{5^3} = (5^3)^{\frac{1}{6}} = 5^{\frac{3}{6}} = 5^{\frac{1}{2}} = \sqrt{5}$.

$\therefore \sqrt[6]{125} = \sqrt{5} = 2.236+$.

Example 2. — $\sqrt[9]{64} = \sqrt[9]{2^6} = (2^6)^{\frac{1}{9}} = 2^{\frac{2}{3}} = \sqrt[3]{2^2} = \sqrt[3]{4}$.

EXERCISE 169

Reduce to radicals of lower order (see § 227):

1. $\sqrt[4]{81}$
2. $\sqrt[4]{256}$
3. $\sqrt[6]{8}$
4. $\sqrt[4]{49}$
5. $\sqrt[6]{16}$
6. $\sqrt[6]{25}$
7. $\sqrt[8]{36}$
8. $\sqrt[8]{900}$
9. $\sqrt[10]{243}$
10. $\sqrt[6]{27}$
11. $\sqrt[8]{64}$
12. $\sqrt[10]{121}$
13. $\sqrt[4]{225}$
14. $\sqrt[6]{144}$
15. $\sqrt[10]{32}$
16. $\sqrt[12]{64}$
17. $\sqrt[9]{125}$
18. $\sqrt[4]{9a^2b^2}$
19. $\sqrt[6]{125a^3b^3}$
20. $\sqrt[10]{243w^5}$
21. $\sqrt[4]{25c^4}$
22. $\sqrt[8]{16m^8y^8}$
23. $\sqrt[6]{27a^3b^3}$
24. $\sqrt[12]{169x^6y^4}$

230. Removing a factor from the radicand.

Example 1. — $\sqrt{75} = \sqrt{25 \cdot 3} = \sqrt{25} \cdot \sqrt{3} = 5\sqrt{3}$.

$\therefore \sqrt{75} = 5\sqrt{3} = 5(1.732) = 8.66$.

Example 2. —
$$\begin{aligned}\sqrt[5]{96a^5b^{12}c^8} &= \sqrt[5]{(32a^5b^{10}c^5)(3b^2c^3)}\\ &= \sqrt[5]{32a^5b^{10}c^5} \cdot \sqrt[5]{3b^2c^3}\\ &= 2ab^2c \cdot \sqrt[5]{3b^2c^3}.\end{aligned}$$

Rule. — **To simplify a radical by removing factors from the radicand:**

1. Separate the radicand into factors as many of which as possible are perfect powers corresponding to the indicated root.

Thus, when it is a *square root,* separate the radicand into as many *perfect square factors* as possible; when it is a *cube root,* separate the radicand into as many *perfect cube factors* as possible; etc.

2. Find the indicated root of each of the perfect powers. Obtain the product of these roots.

3. Multiply the result of Step 2 by the indicated root of the remaining factors of the radicand.

Example 3. — $\sqrt[7]{128\,x^3y^8} = \sqrt[7]{2^7 \cdot y^7 \cdot x^3y} = 2\,y\sqrt[7]{x^3y}.$

EXERCISE 170

Simplify by removing factors from the radicand (see § 227):

1. $\sqrt{18}$

2. $\sqrt{27}$

3. $\sqrt{200}$

4. $\sqrt{125}$

5. $\sqrt{48}$

6. $\sqrt{75}$

7. $\sqrt{63\,m^2}$

8. $\sqrt{28\,a^2b}$

9. $\sqrt{12\,x^3y^2}$

10. $\sqrt{80\,c^4d^3}$

11. $\sqrt{45\,m^4n^3}$

12. $\sqrt{32\,x^3y^5}$

13. $\sqrt{99\,ab^5c^3}$

14. $\sqrt{\frac{1}{75}\,m^3n^2}$

15. $\sqrt{\frac{1}{60}\,x^2y^7}$

16. $\sqrt{98\,rs^3}$

17. $\sqrt{96\,m^2n^4}$

18. $\sqrt{108\,a^7}$

19. $\sqrt{300\,m^5n^2}$

20. $\sqrt{363\,x^6}$

21. $\sqrt[3]{24\,x^3}$

22. $\sqrt[3]{54\,y^5}$

23. $\sqrt[3]{128\,t^6}$

24. $\sqrt[4]{32\,m^4}$

25. $\sqrt[4]{162\,a^2b^4}$

26. $\sqrt[4]{80\,c^3d^5}$

27. $\sqrt[5]{32\,x^5y^6}$

28. $\sqrt[6]{64\,x^7y}$

29. $\sqrt[3]{250\,a^3b^4}$

30. $\sqrt[3]{16\,x^3m^2}$

31. $\sqrt[3]{72\,ab^4}$

32. $\sqrt[5]{243\,x^2m}$

33. $\sqrt{\dfrac{60\,x^2}{9\,y^4}}$

34. $\sqrt{\dfrac{99\,mn^2}{16\,p^4}}$

35. $\sqrt[3]{\dfrac{4\,a^3}{27\,x^{12}y^6}}$

36. $\sqrt[4]{\dfrac{2\,a}{81\,x^8y^4}}$

37. $\sqrt[3]{\dfrac{8\,xy}{125\,a^6m^3}}$

38. $\sqrt[4]{\dfrac{5\,x^4y^8}{256}}$

39. $\sqrt[5]{\dfrac{3\,x^{15}}{a^{10}y^5}}$

40. $\sqrt[5]{\dfrac{64\,y^5}{x^{10}}}$

231. Changing a fractional to an integral radicand.

Example 1. — $\sqrt{\dfrac{5}{12\,a}} = \sqrt{\dfrac{5\cdot 3\,a}{12\,a\cdot 3\,a}} = \sqrt{\dfrac{15\,a}{36\,a^2}} = \dfrac{1}{6\,a}\sqrt{15\,a}.$

Observe that $12\,a$ is multiplied by $3\,a$ to change it into $36\,a^2$, the "smallest" perfect square which is a multiple of $12\,a$. Then the numerator is also multiplied by $3\,a$ to avoid changing the value of the fraction.

Rule. — To change a fractional to an integral radicand:

1. Multiply both terms of the fraction by the number which will make the denominator the least perfect power of degree corresponding to the order of the radical.

2. Simplify the resulting radical as in § 230.

NOTE 1. — This process is called *rationalizing the denominator.*

NOTE 2. — If it is a *cube root,* the denominator must be made a *perfect cube; if it is a fourth root, it must be made a perfect fourth power;* etc.

Example 2. — $\sqrt[3]{\frac{27}{4\,a^4}} = \sqrt[3]{\frac{27 \cdot 2\,a^2}{4\,a^4 \cdot 2\,a^2}} = \sqrt[3]{\frac{3^3 \cdot 2 \cdot a^2}{2^3 \cdot a^6}} = \frac{3}{2\,a^2}\sqrt[3]{2\,a^2}.$

EXERCISE 171

Express with integral radicands (see § 227):

1. $\sqrt{\frac{2}{5}}$
2. $\sqrt{\frac{3}{5}}$
3. $\sqrt{\frac{5}{7}}$
4. $\sqrt{\frac{7}{12}}$
5. $\sqrt{\frac{11}{32}}$
6. $\sqrt{\frac{5}{24}}$
7. $\sqrt{\frac{3}{7\,x}}$
8. $\sqrt{\frac{3\,a}{8\,b}}$
9. $\sqrt{\frac{5\,x}{18\,y}}$
10. $\sqrt{\frac{3\,m}{11\,n}}$
11. $\sqrt{\frac{5\,r^3}{8\,s^3}}$
12. $\sqrt{\frac{3\,c^2}{20\,t}}$
13. $\sqrt{\frac{xy}{200}}$
14. $\sqrt{\frac{13}{50\,cd}}$
15. $\sqrt{\frac{11}{48\,m^2}}$
16. $\sqrt[3]{\frac{2}{3}}$
17. $\sqrt[3]{\frac{7}{5}}$
18. $\sqrt[3]{\frac{3\,a}{2\,c}}$
19. $\sqrt[3]{\frac{5}{16}}$
20. $\sqrt[3]{\frac{2}{25}}$
21. $\sqrt[3]{\frac{5\,a^3}{2\,x^5}}$
22. $\sqrt[3]{\frac{2\,a^2}{3\,b^2}}$
23. $\sqrt[4]{\frac{3}{8\,x^3}}$
24. $\sqrt[4]{\frac{5\,m}{27\,n^3}}$
25. $\sqrt[4]{\frac{bx}{a}}$
26. $\sqrt[5]{\frac{5\,a^2}{16}}$
27. $\sqrt{\frac{2\,m^{11}}{9\,a^9}}$
28. $\sqrt[6]{\frac{a^2y^7}{32\,x^6}}$
29. $\sqrt{\frac{x^2y}{x-y}}$
30. $\sqrt{\frac{c-d}{c+d}}$

232. **To reduce a radical to its simplest form:**

a. Reduce it to an equivalent radical of as low order as possible, as in § 229;

b. Remove from the radicand any perfect power factors of the radicand, as in § 230;

c. Change the radicand to an integral radicand, as in § 231.

233. **Addition and subtraction of radicals.**

The sum of two radicals like $\sqrt{2}$ and $\sqrt{3}$ can be indicated; thus $\sqrt{2} + \sqrt{3}$. Or, each can be expressed decimally and the approximate sum be obtained; thus,

$$\sqrt{2} + \sqrt{3} = 1.414 + 1.732 = 3.146.$$

Similarly the sum of $\sqrt[3]{2}$ and $\sqrt{2}$ is $\sqrt[3]{2} + \sqrt{2}$.

In order that radicals may be combined, they must be similar radicals.

Similar radicals are radicals of the same order, which have the same radicand.

Thus, $2\sqrt{13a}$ and $3\sqrt{13a}$ are similar radicals. They differ only in their coefficients.

Also, $\sqrt[3]{ax^2}$ and $\sqrt[3]{ax^2}$ are similar radicals;
but $\sqrt[3]{a^2x}$ and $\sqrt[3]{ax^2}$ are not similar radicals.

Rule. — To add or subtract radicals:

1. Change them to equivalent radicals of simplest form. (§ 232)

2. Combine similar radicals; indicate the addition of dissimilar radicals.

Example. —
$$\begin{aligned}\sqrt[3]{\tfrac{1}{4}} - \sqrt[3]{24} + \sqrt[3]{54} &= \sqrt[3]{\tfrac{2}{8}} - \sqrt[3]{8 \cdot 3} + \sqrt[3]{27 \cdot 2}\\ &= \tfrac{1}{2}\sqrt[3]{2} - 2\sqrt[3]{3} + 3\sqrt[3]{2}\\ &= 3\tfrac{1}{2}\sqrt[3]{2} - 2\sqrt[3]{3}.\end{aligned}$$

EXERCISE 172

Simplify by combining like radicals:

1. $\sqrt{32} + \sqrt{72}$
2. $2\sqrt{18x} - \sqrt{50x}$
3. $\sqrt{128a} - 3\sqrt{8a}$
4. $\sqrt{\frac{1}{8}} + \frac{1}{4}\sqrt{18}$
5. $\sqrt{\frac{75}{2}m} - \sqrt{\frac{27}{8}m}$
6. $\sqrt{\frac{8}{3}x^3} + 5\sqrt{\frac{32}{3}x^3}$
7. $3a\sqrt{16a} - \sqrt{144a^3}$
8. $2x\sqrt{25xy^2} - 3y\sqrt{16x^3}$
9. $n\sqrt{98m^3n} - m\sqrt{32mn^3}$
10. $\sqrt[3]{16} + \sqrt[3]{250}$
11. $\sqrt[3]{54x} - \sqrt[3]{16x}$
12. $\sqrt[3]{192y^2} - \sqrt[3]{24y^2}$
13. $\sqrt[3]{\frac{2}{3}} + \sqrt[3]{\frac{9}{4}}$
14. $\frac{5}{6}\sqrt[3]{\frac{1}{5}a} - \sqrt[3]{\frac{5}{54}a}$
15. $3x\sqrt[3]{\frac{1}{9}x} + 5\sqrt[3]{3x^4}$
16. $\sqrt[4]{81a^5} + \sqrt[4]{256a^5}$
17. $\sqrt[5]{729} - \sqrt[5]{3}$
18. $\sqrt[6]{2} + \sqrt[6]{128}$

19. $\sqrt{\dfrac{a-1}{a+1}}+\sqrt{\dfrac{a+1}{a-1}}-\dfrac{2}{a^2-1}\sqrt{a^2-1}$

20. $x^2y^2\sqrt[6]{\dfrac{2}{x^5y^5}}+\sqrt[6]{2\,x^7y^7}$ 21. $\sqrt[10]{x^5}+\sqrt{x}$

MULTIPLICATION AND DIVISION OF RADICALS

234. Changing radicals of different orders to equivalent radicals of the same order is necessary before radicals can be multiplied or divided.

Example. — Change $\sqrt{2}$, $\sqrt[3]{3}$, and $\sqrt[4]{5}$ to radicals of the same order; and determine which is the greatest number.

Solution. — 1. $\sqrt{2}=2^{\frac{1}{2}}$; $\sqrt[3]{3}=3^{\frac{1}{3}}$; $\sqrt[4]{5}=5^{\frac{1}{4}}$.

2. Change the fractional exponents to their L. C. D.

$$2^{\frac{1}{2}}=2^{\frac{6}{12}};\quad 3^{\frac{1}{3}}=3^{\frac{4}{12}};\quad 5^{\frac{1}{4}}=5^{\frac{3}{12}}.$$

3 $$\left.\begin{aligned}2^{\frac{1}{2}}&=2^{\frac{6}{12}}=\sqrt[12]{2^6}=\sqrt[12]{64}\\3^{\frac{1}{3}}&=3^{\frac{4}{12}}=\sqrt[12]{3^4}=\sqrt[12]{81}\\5^{\frac{1}{4}}&=5^{\frac{3}{12}}=\sqrt[12]{5^3}=\sqrt[12]{125}\end{aligned}\right\}\quad\therefore\ \sqrt[4]{5}\text{ is the largest number.}$$

Rule. — To change radicals of several orders to equivalent radicals of the same order:

1. **Express the radicals by means of fractional exponents.**
2. **Change the exponents to their lowest common denominator.**
3. **Rewrite the resulting expressions with radical signs.**

EXERCISE 173

Change the radicals in each example to radicals of the same order; select the greatest in Examples 1, 2, 7–12.

1. $\sqrt[3]{3}$ and $\sqrt{2}$
2. $\sqrt[4]{4}$ and $\sqrt[6]{5}$
3. $\sqrt[3]{4\,a}$ and $\sqrt[4]{6\,a^2}$
4. $\sqrt[3]{11\,b}$ and $\sqrt[6]{124\,b}$
5. $\sqrt{8\,x^3}$ and $\sqrt[3]{7\,x^4}$
6. $\sqrt[5]{a^2b^2}$ and $\sqrt{ab}$
7. $\sqrt{5}$ and $\sqrt[6]{124}$
8. $\sqrt[3]{\frac{1}{4}}$ and $\sqrt{\frac{1}{2}}$
9. $\sqrt[6]{3}$, $\sqrt[3]{2}$, and $\sqrt[4]{5}$
10. $\sqrt[6]{6}$, $\sqrt[3]{2}$, and $\sqrt{3}$
11. $\sqrt{2}$, $\sqrt[4]{5}$, and $\sqrt[3]{3}$
12. $\sqrt{2}$, $\sqrt[3]{4}$, and $\sqrt[4]{8}$

13. $\sqrt[3]{c+d}$ and $\sqrt{c+d}$

14. $\sqrt[3]{x-y}$ and $\sqrt{x+y}$

15. $\sqrt[3]{a-2}$ and $\sqrt[4]{a+2}$

16. $\sqrt[3]{m+n}$ and $\sqrt[5]{m-n}$

235. Multiplication of radicals. In § 224, we have assumed that $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} \cdot b^{\frac{1}{n}}$, or $a^{\frac{1}{n}} \cdot b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}$.

Written with radical signs, this becomes $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$.

That is, the product of the *n*th roots of two numbers equals the *n*th root of the product of the two numbers.

Thus, $2\sqrt{3} \cdot 3\sqrt{6} = 6\sqrt{3 \cdot 6} = 6\sqrt{3^2 \cdot 2} = 6 \cdot 3\sqrt{2} = 18\sqrt{2}$.

$$4\sqrt[3]{2} \cdot 5\sqrt[3]{4} = 20\sqrt[3]{8} = 20 \cdot 2 = 40.$$

$$\sqrt{2} \cdot \sqrt[3]{2} = 2^{\frac{1}{2}} \cdot 2^{\frac{1}{3}} = 2^{\frac{3}{6}} \cdot 2^{\frac{2}{6}} = \sqrt[6]{2^3} \cdot \sqrt[6]{2^2} = \sqrt[6]{2^5}.$$

Rule. — To multiply radicals:

1. Change the radicals, if necessary, to radicals of the same order. (§ 234)

2. Place the product of the radicands obtained in Step 1 under the common root.

3. Reduce the result of Step 2 to simplest form.

EXERCISE 174

Find the following products (see § 227):

1. $\sqrt{3} \cdot \sqrt{18}$
2. $\sqrt{2} \cdot \sqrt{14}$
3. $2\sqrt{7} \cdot \sqrt{21}$
4. $3\sqrt{6} \cdot 2\sqrt{12}$
5. $3\sqrt{3} \cdot \sqrt{27}$
6. $5\sqrt{20} \cdot \sqrt{30}$
7. $3\sqrt{7} \cdot 5\sqrt{7}$
8. $2\sqrt{12} \cdot 3\sqrt{3}$
9. $(\sqrt{5})^2$
10. $(2\sqrt{3})^2$
11. $(4\sqrt{5})^2$
12. $(6\sqrt{2})^2$
13. $2\sqrt{6a} \cdot 5\sqrt{3a}$
14. $2\sqrt{6x} \cdot 3\sqrt{8x}$
15. $(\sqrt{a+b})^2$
16. $(\sqrt{x-3})^2$
17. $(3\sqrt{y+2})^2$
18. $(4\sqrt{z-5})^2$

19. Multiply $2\sqrt{3} + 3\sqrt{2}$ by $3\sqrt{3} - \sqrt{2}$.

Solution. —

$$\begin{array}{l} 2\sqrt{3} + 3\sqrt{2} \\ \underline{3\sqrt{3} - \sqrt{2}} \\ 18 + 9\sqrt{6} \\ \underline{\qquad - 2\sqrt{6} - 6} \\ 18 + 7\sqrt{6} - 6 = 12 + 7\sqrt{6} \end{array}$$

$\therefore\ (2\sqrt{3} + 3\sqrt{2})(3\sqrt{3} - \sqrt{2}) = 12 + 7(2.44) = 12 + 17.08 = 29.08.$

Find similarly the following products:

20. $(2-\sqrt{5})(2+\sqrt{5})$

✓ **21.** $(7-\sqrt{3})(6+\sqrt{3})$

22. $(\sqrt{5}+4)(\sqrt{5}-3)$

23. $(2\sqrt{5}+7)(3\sqrt{5}-7)$

24. $(\sqrt{5}-4)^2$

25. $(2\sqrt{2}-7)^2$

26. $(2\sqrt{2}+3)^2$

27. $(2\sqrt{2}+3)(4\sqrt{2}-5)$

28. $(3\sqrt{3}-5)(3\sqrt{3}+5)$

29. $(\sqrt{2a}-\sqrt{x})(\sqrt{2a}+\sqrt{x})$

30. $(\sqrt{15}-\sqrt{3})(\sqrt{15}+\sqrt{6})$

31. $\sqrt{x+1}\cdot\sqrt{x+1}$

32. $(\sqrt{a-1}+1)^2$

33. $(\sqrt{x-5}-3)^2$

34. $(\sqrt{y}-\sqrt{y+3})^2$

35. $(\sqrt{x+3}+\sqrt{x-5})^2$

36. Find $\sqrt[3]{9a^2}\cdot\sqrt{15a}$.

Solution. — 1. $\sqrt[3]{9a^2}\cdot\sqrt{15a}=(3^2\cdot a^2)^{\frac{1}{3}}\cdot(3\cdot 5a)^{\frac{1}{2}}$

2. $=(3^2\cdot a^2)^{\frac{2}{6}}\cdot(3\cdot 5\cdot a)^{\frac{3}{6}}$

3. $=\sqrt[6]{3^4\cdot a^4\cdot 3^3\cdot 5^3\cdot a^3}.$

(Complete the solution.)

Find:

37. $\sqrt[3]{9}\cdot\sqrt[3]{6}$

38. $3\sqrt[3]{16}\cdot\sqrt[3]{36}$

39. $\sqrt[4]{3xy}\cdot\sqrt[4]{27x^2y^3}$

40. $\sqrt[3]{54x^2}\cdot\sqrt[3]{9xy}$

41. $3\sqrt[5]{4a^2}\cdot 2\sqrt[5]{40a^3}$

42. $3\sqrt[6]{27x^5}\cdot\sqrt[6]{81x^2}$

✓ **43.** $\sqrt{2a}\cdot\sqrt[3]{4a^2}$

44. $\sqrt{x}\cdot\sqrt[4]{x^3}$

45. $\sqrt[3]{2m^2}\cdot\sqrt[4]{8m^3}$

46. $\sqrt{10}\cdot\sqrt[3]{25}$

47. $\sqrt[3]{16rt^2}\cdot\sqrt{4t}$

48. $2a\sqrt{x}\cdot 5\sqrt[4]{4x}$

49. $\sqrt[3]{x^2y^2}\cdot\sqrt[6]{2x^4y^3}$

50. $\sqrt[4]{8c^3}\cdot\sqrt[3]{4c^2}$

51. $\sqrt[3]{9x^2y}\cdot\sqrt{15xy^3}$

52. $\sqrt[3]{12m^2}\cdot\sqrt[6]{8m^3}$

236. Division of monomial radicals. (See § 227.)

Consider $\sqrt{6}\div\sqrt{2}$. It would be possible to obtain the approximate value of the result by using the approximate decimal values of $\sqrt{6}$ and $\sqrt{2}$.

Thus,
$$\frac{\sqrt{6}}{\sqrt{2}}=\frac{2.449}{1.414}=1.7319.$$

This solution requires the computation of two square roots, and the division of one decimal by another.

A preferable solution follows:

$$\frac{\sqrt{6}}{\sqrt{2}} = \sqrt{\frac{6}{2}} = \sqrt{3} = 1.732.$$

This solution requires only one square root, the division of a decimal by an integer, and is more accurate.

Similarly, $$\frac{\sqrt[3]{8}}{\sqrt[3]{18}} = \sqrt[3]{\frac{8}{18}} = \sqrt[3]{\frac{4}{9}} = \sqrt[3]{\frac{12}{27}} = \frac{1}{3}\sqrt[3]{12}.$$

Rule. — To divide one monomial radical by another:

1. Change the radicals, if necessary, to equivalent radicals of the same order.

2. Write the quotient of the radicands obtained in Step 2 under the common root.

3. Simplify the radical resulting in Step 2, by the methods summarized in § 232.

EXERCISE 175

Perform the indicated divisions (see § 227):

1. $\sqrt{27} \div \sqrt{9}$

2. $\sqrt{21} \div \sqrt{7}$

3. $9\sqrt{20} \div 3\sqrt{5}$

4. $\sqrt{50\,x} \div \sqrt{2\,x}$

5. $3\,a\sqrt{90} \div a\sqrt{3}$

6. $\sqrt{3} \div \sqrt{\frac{1}{3}}$

7. $20\sqrt{x^3y^3} \div 5\sqrt{xy}$

8. $14\sqrt{30\,a^3} \div 7\sqrt{6\,a}$

9. $4\,x\sqrt{24\,x^3m} \div 8\,x\sqrt{8\,x}$

10. $\sqrt{14\,y^3} \div \sqrt{\frac{1}{7}\,y^2}$

11. $121\sqrt{cd} \div 11\sqrt{\frac{1}{4}\,d}$

12. $\sqrt{2} \div \sqrt{3}$

13. $\sqrt{5} \div \sqrt{6}$

14. $\sqrt{8\,t} \div \sqrt{7\,t}$

15. $\sqrt{33\,m^2n} \div \sqrt{18\,n}$

16. $(6\sqrt{15} - 8\sqrt{12}) \div 2\sqrt{3}$

17. $(5\sqrt{x} + 20\sqrt{x}) \div 5\sqrt{x}$

18. $(\sqrt{6} + 2\sqrt{12}) \div \sqrt{3}$

19. $(4\sqrt{12} - 3\sqrt{18}) \div \sqrt{6}$

20. $(5\sqrt{20} - \sqrt{15}) \div \sqrt{5}$

21. $\sqrt[3]{250} \div \sqrt[3]{2}$

22. $\sqrt[3]{84} \div \sqrt[3]{21}$

23. $\sqrt[5]{39\,x^2} \div \sqrt[5]{13\,x^6}$

24. $\sqrt[4]{48\,x^8} \div \sqrt[4]{3\,x^5}$

25. $\sqrt[6]{125\,a^2b^2} \div \sqrt[6]{25\,b^2c}$

26. $4\,x\sqrt[3]{12\,x^2y} \div x\sqrt[3]{6\,xy^3}$

27. $3\,m\sqrt[4]{12\,s^3} \div 12\,m\sqrt[4]{\frac{1}{4}\,s^3}$

28. $5\,c^2\sqrt[3]{6\,cd} \div \frac{1}{15}\sqrt[3]{\frac{1}{9}\,c^2d}$

29. $\sqrt[m]{8\,x^3y^2} \div \sqrt[m]{4\,xy^2}$

30. $\sqrt[n]{ax^2y^3} \div \sqrt[n]{axy^2}$

31. Find $\sqrt{3} \div \sqrt[3]{9}$

Solution. — $\dfrac{\sqrt{3}}{\sqrt[3]{9}} = \dfrac{3^{\frac{1}{2}}}{(3^2)^{\frac{1}{3}}} = \dfrac{3^{\frac{3}{6}}}{(3^2)^{\frac{2}{6}}} = \dfrac{\sqrt[6]{3^3}}{\sqrt[6]{3^4}} = ?$

(Complete the solution.)

32. $3 \div \sqrt[3]{3}$

33. $2 \div \sqrt[4]{2}$

34. $12 \div \sqrt[3]{3}$

35. $6 \div \sqrt[4]{36}$

36. $x \div \sqrt[6]{x}$

37. $\sqrt{3} \div \sqrt[3]{3}$

38. $\sqrt{2} \div \sqrt[5]{2}$

39. $\sqrt[3]{4} \div \sqrt[6]{8}$

40. $\sqrt{16\,a} \div \sqrt[4]{32\,a}$

41. $\sqrt[3]{18\,x^2} \div \sqrt{6\,x}$

42. $\sqrt[6]{\frac{4}{25}} \div \sqrt[3]{\frac{2}{5}}$

43. $\sqrt[3]{9\,x^2y} \div \sqrt[6]{81\,xy}$

44. $10\sqrt[3]{216} \div 2\sqrt{6}$

45. $3\sqrt[6]{90\,x^2y} \div 2\sqrt[3]{6\,xy^2}$

237. Division by a binomial quadratic surd.

Consider $\dfrac{1}{2+\sqrt{3}}$. The approximate result can be found in decimal form as follows:

$$\frac{1}{2+\sqrt{3}} = \frac{1}{2+1.732} = \frac{1}{3.732} = .267^{+}$$

This solution requires the division of an integer by a decimal.

A preferable solution follows:

We observe that $2+\sqrt{3}$ multiplied by $2-\sqrt{3} = 4-3 = 1$. Multiply both terms of the fraction by $(2-\sqrt{3})$:

$$\text{then } \frac{1}{2+\sqrt{3}} = \frac{2-\sqrt{3}}{(2+\sqrt{3})(2-\sqrt{3})} = \frac{2-\sqrt{3}}{4-1} = \frac{2-\sqrt{3}}{1}$$

$$\therefore \frac{1}{2+\sqrt{3}} = 2 - 1.732^{+} = .267^{+}.$$

In this solution, *division by a decimal is avoided.* The denominator is multiplied by a number such that the product is a *rational number.* For this reason, **this process is called rationalizing the denominator of the fraction.**

If the denominator is of the form $a + b\sqrt{c}$, it is multiplied by $a - b\sqrt{c}$. These two surd expressions are called **conjugate surds.** Observe $(a + b\sqrt{c})(a - b\sqrt{c}) = a^2 - b^2\sqrt{c^2}$.

$$\therefore (a + b\sqrt{c})(a - b\sqrt{c}) = a^2 - b^2c.$$

Their product is a rational number.

Rule. — To divide a number by a binomial quadratic surd, multiply dividend and divisor by the conjugate of the divisor, and simplify the result.

Example. — Rationalize the denominator of $\dfrac{3\sqrt{2}+1}{2\sqrt{2}-1}$.

$$\frac{3\sqrt{2}+1}{2\sqrt{2}-1} = \frac{(3\sqrt{2}+1)(2\sqrt{2}+1)}{(2\sqrt{2}-1)(2\sqrt{2}+1)} = \frac{12+5\sqrt{2}+1}{8-1}$$

$$\therefore \frac{3\sqrt{2}+1}{2\sqrt{2}-1} = \frac{13+5\sqrt{2}}{7} = \frac{13+5(1.414^+)}{7} = 2.86.$$

EXERCISE 176

Perform the following divisions by rationalizing the divisor:

1. $\dfrac{10}{3+\sqrt{7}}$
2. $\dfrac{3}{\sqrt{8}+4}$
3. $\dfrac{4}{9-\sqrt{2}}$
4. $\dfrac{3}{5-2\sqrt{2}}$
5. $\dfrac{5}{2\sqrt{5}-3}$
6. $\dfrac{6}{3+2\sqrt{5}}$
7. $\dfrac{5+\sqrt{3}}{2-\sqrt{3}}$
8. $\dfrac{2+\sqrt{2}}{4-\sqrt{2}}$
9. $\dfrac{\sqrt{6}-\sqrt{3}}{\sqrt{6}+\sqrt{3}}$
10. $\dfrac{\sqrt{3}+\sqrt{5}}{\sqrt{3}-\sqrt{5}}$
11. $\dfrac{\sqrt{10}}{\sqrt{5}-\sqrt{2}}$
12. $\dfrac{\sqrt{x}-3}{\sqrt{x}+3}$
13. $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$
14. $\dfrac{2\sqrt{3}-5}{\sqrt{3}+5}$
15. $\dfrac{\sqrt{6}+2\sqrt{3}}{2\sqrt{6}-\sqrt{3}}$

16. $\dfrac{2\sqrt{2}+5}{\sqrt{3}-2}$

17. $\dfrac{3+\sqrt{2}}{3-\sqrt{2}}$

18. $\dfrac{c-a\sqrt{b}}{c+a\sqrt{b}}$

19. $\dfrac{\sqrt{a-1}+2}{\sqrt{a-1}+1}$

20. $\dfrac{\sqrt{x+y}-\sqrt{x-y}}{\sqrt{x+y}+\sqrt{x-y}}$

238. **Involution and evolution of monomial surds.**

Example 1. — $(\sqrt[6]{12})^3 = (12^{\frac{1}{6}})^3 = 12^{\frac{3}{6}} = 12^{\frac{1}{2}} = \sqrt{12} = 2\sqrt{3}$

$\therefore\ (\sqrt[6]{12})^3 = 2\sqrt{3} = 2(1.732^+) = 3.464^+$

Example 2. — $\sqrt[3]{\sqrt[5]{27\,x^3}} = \{(3^3 \cdot x^3)^{\frac{1}{5}}\}^{\frac{1}{3}} = (3^3 \cdot x^3)^{\frac{1}{15}} = 3^{\frac{1}{5}}x^{\frac{1}{5}}$

$\therefore\ \sqrt[3]{\sqrt[5]{27\,x^3}} = \sqrt[5]{3\,x}$

EXERCISE 177

Simplify the following expressions:

1. $(\sqrt[4]{7})^2$
2. $(\sqrt[6]{27})^2$
3. $(\sqrt[8]{16})^3$
4. $(\sqrt[8]{12})^4$
5. $(\sqrt[3]{125})^4$
6. $(2\,x\sqrt[3]{a^2})^4$
7. $(y\sqrt[5]{3\,x})^5$
8. $\sqrt{4\sqrt{x}}$
9. $\sqrt[3]{\sqrt{a^3b^3c^6}}$
10. $\sqrt{\sqrt[3]{2\,x^7}}$
11. $\sqrt[4]{\sqrt[3]{36\,m^6}}$
12. $\sqrt[7]{\sqrt{128}}$
13. $\sqrt{\sqrt[6]{36}}$
14. $\sqrt[3]{\sqrt{125}}$
15. $\sqrt[5]{\sqrt[3]{243\,a^{10}}}$
16. $\sqrt{\sqrt[4]{9\,a^{16}}}$
17. $\sqrt[3]{\sqrt[6]{27\,a^3b^6}}$
18. $(\sqrt[8]{a^5b^{11}c^5})^4$
19. $(2\,a\sqrt[3]{a^2})^3$
20. $(3\,m^2\sqrt[4]{m^3})^3$
21. $\sqrt[4]{\sqrt[3]{16\,x^4y^2}}$
22. $\sqrt{\sqrt[3]{a^2-8\,a+16}}$
23. $\sqrt{\sqrt[4]{4\,x^2-4\,xy+y^2}}$
24. $(\sqrt[6]{3\,x-2})^3$
25. $(\sqrt[4]{(2\,r-5\,s)^2})^2$

239. **The square roots of a binomial surd expression of the form $a+\sqrt{b}$ can often be found by inspection.**

Preparation. — Observe that $(\sqrt{2}-\sqrt{3})^2 = 2 - 2\sqrt{6} + 3 = 5 - 2\sqrt{6}$.

Notice that the square of a binomial surd is a binomial surd; that 5 is the sum of the two radicands 2 and 3, and that 6 is the product of the two radicands 2 and 3. This suggests the

Rule. — To find the square root of a binomial surd expression of the form $a + \sqrt{b}$:

1. Change the surd part so that its coefficient is 2.

2. If possible, separate the rational term into two numbers whose product shall be the radicand obtained in Step 1.

3. If Step 2 has been possible, extract the principal square root of each of the numbers obtained in Step 2, and connect them by the sign of the surd term.

Example. — Find the square root of $22 - 3\sqrt{32}$.

Solution. — 1. $22 - 3\sqrt{32} = 22 - \sqrt{9 \cdot 8 \cdot 4}$ (See Note below.)

2. $\therefore\ 22 - 3\sqrt{32} = 22 - 2\sqrt{72}$

3. $22 = 18 + 4;\quad 72 = 18 \cdot 4$

4. $\therefore\ \sqrt{22 - 3\sqrt{32}} = \sqrt{22 - 2\sqrt{72}}$

5. $= \sqrt{18 + 4 - 2\sqrt{72}}$

6. $= \sqrt{18} - \sqrt{4}$

7. $= 3\sqrt{2} - 2$

Check. — $(3\sqrt{2} - 2)^2 = 18 - 12\sqrt{2} + 4$

$= 22 - 3 \cdot 4\sqrt{2}$, or $22 - 3\sqrt{32}$.

NOTE 1. — Observe the change from $3\sqrt{32}$ to $\sqrt{9 \cdot 8 \cdot 4}$. 3 before the radical is the same as $\sqrt{9}$; and $\sqrt{9} \cdot \sqrt{32} = \sqrt{9 \cdot 8 \cdot 4}$. When a coefficient of a square root is placed under the radical, it must be squared. Thus, $a\sqrt{5} = \sqrt{5a^2}$; $6\sqrt{x} = \sqrt{36x^2}$.

NOTE 2. — $[-(3\sqrt{2} - 2)]$ or $[2 - 3\sqrt{2}]$ also is a square root of $22 - 2\sqrt{72}$.

EXERCISE 178

Find the square root of:

1. $5 + 2\sqrt{6}$

2. $8 - 2\sqrt{15}$

3. $12 + 2\sqrt{35}$

4. $8 - \sqrt{48}$

5. $14 + \sqrt{132}$

6. $11 - 4\sqrt{6}$

7. $10 - \sqrt{84}$

8. $13 + 4\sqrt{10}$

9. $12 - 8\sqrt{2}$

IMAGINARY NUMBERS

240. An introduction to imaginary numbers was given in § 189, page 285. Review this section.

241. Powers of the imaginary unit i.

Since $i = -1$, then $\qquad i^2 = -1$

$i^3 = i^2 \cdot i = -1 \cdot i \qquad \therefore\ i^3 = -i$

$i^4 = i^2 \cdot i^2 = (-1) \cdot (-1) \quad \therefore\ i^4 = +1$

Therefore the first four powers of i are i, -1, $-i$, and 1.

These four values recur in regular order; thus, $i^5 = i$; $i^6 = -1$; $i^7 = -i$; and $i^8 = 1$. (Prove these statements.)

242. Multiplication of imaginary numbers and complex numbers. (See § 190.)

Example 1. — $\sqrt{-2} \cdot \sqrt{-3} = i\sqrt{2} \cdot i\sqrt{3} = i^2\sqrt{6} = -6$

Example 2. — Find $(2 - \sqrt{-3})(5 + \sqrt{-3})$.

Solution. — 1. $(2 - \sqrt{-3})(5 + \sqrt{-3}) = (2 - i\sqrt{3})(5 + i\sqrt{3})$

2. $= 10 - 3\,i\sqrt{3} - i^2 \cdot 3$

3. $= 10 - 3\,i\sqrt{3} - (-1) \cdot 3$

4. $= 13 - 3\,i\sqrt{3}$.

EXERCISE 179

Find the following products:

1. $\sqrt{-4} \cdot \sqrt{-16}$
2. $\sqrt{-2} \cdot \sqrt{-18}$
3. $\sqrt{-9} \cdot \sqrt{-2}$
4. $\sqrt{-5} \cdot \sqrt{-10}$
5. $\sqrt{-25\,x^2} \cdot \sqrt{-4\,x^2}$
6. $3\sqrt{-5} \cdot 3\sqrt{-5}$
7. $4\sqrt{-3} \cdot 6\sqrt{-3}$
8. $x\sqrt{-y} \cdot x\sqrt{-y}$
9. $a\sqrt{-b} \cdot c\sqrt{-d}$
10. $\sqrt{-xy} \cdot (-\sqrt{-ab})$
11. $(3 + \sqrt{-2})(3 - \sqrt{-2})$
12. $(2 - \sqrt{-6})(2 + \sqrt{-6})$
13. $(2 + \sqrt{-5})(2 + 3\sqrt{-5})$
14. $(8 - \sqrt{-7})(10 + \sqrt{-7})$
15. $(-2 + \sqrt{-2})(-2 - \sqrt{-2})$
16. $(6 - \sqrt{-3})^2$
17. $(a + \sqrt{-b})^2$
18. $\{\frac{1}{3}(-1 + \sqrt{-2})\}^2$
19. $\{\frac{1}{2}(-1 - \sqrt{-3})\}^2$
20. $\{\frac{1}{2}(-1 + \sqrt{-3})\}^3$

243. **Division of imaginary numbers can be accomplished in two ways.**

Example 1. — $$\frac{\sqrt{-12}}{\sqrt{-3}} = \frac{i\sqrt{12}}{i\sqrt{3}} = \sqrt{4} = 2.$$

In this case, the division is performed directly.

Example 2. — $$\frac{10}{\sqrt{-2}} = \frac{10}{i\sqrt{2}} = \frac{10 \cdot i \cdot \sqrt{2}}{i \cdot \sqrt{2} \cdot i \cdot \sqrt{2}}$$
$$= \frac{10\,i\sqrt{2}}{i^2 \cdot 2} = \frac{5\,i\sqrt{2}}{-1} = -5\,i\sqrt{2}.$$

In this solution, **the denominator has been rationalized by multiplying it by itself.**

Example 3. — Perform the indicated division in $\frac{2}{1+\sqrt{-3}}$ by rationalizing the denominator.

Solution. — 1. $$\frac{2}{1+\sqrt{-3}} = \frac{2}{1+i\sqrt{3}}$$

2. $$= \frac{2(1-i\sqrt{3})}{(1+i\sqrt{3})(1-i\sqrt{3})} = \frac{2(1-i\sqrt{3})}{1-3\,i^2}$$

3. $$= \frac{2(1-i\sqrt{3})}{1+3} = \frac{1-i\sqrt{3}}{2}.$$

Observe that the denominator was rationalized by multiplying it by its *conjugate*. Similarly, to rationalize $3 - i\sqrt{5}$, multiply it by $3 + i\sqrt{5}$; the product will be $9 - i^2\sqrt{25} = 9 + 5$ or 14.

EXERCISE 180

Find the following quotients, or rationalize the denominators:

1. $\frac{\sqrt{-36}}{\sqrt{-6}}$

2. $\frac{4\sqrt{-48}}{2\sqrt{-3}}$

3. $\frac{\sqrt{-32}}{8}$

4. $\frac{\sqrt{-42}}{\sqrt{-7}}$

5. $\frac{\sqrt{-40\,x}}{\sqrt{-5\,x}}$

6. $\frac{6\sqrt{-39\,xy}}{3\sqrt{3\,y}}$

7. $\frac{\sqrt{10}}{\sqrt{-2}}$

8. $\frac{\sqrt{15}}{\sqrt{-3}}$

9. $\frac{\sqrt{28}}{\sqrt{-14}}$

10. $\frac{6}{\sqrt{-3}}$

11. $\frac{14}{\sqrt{-2}}$

12. $\frac{24}{\sqrt{-8}}$

13. $\dfrac{3}{1+\sqrt{-3}}$

14. $\dfrac{13}{3-\sqrt{-2}}$

15. $\dfrac{4}{2-\sqrt{-2}}$

16. $\dfrac{5}{1+\sqrt{-2}}$

17. $\dfrac{1-\sqrt{-1}}{1+2\sqrt{-1}}$

18. $\dfrac{3-\sqrt{-2}}{3-2\sqrt{-1}}$

19. $\dfrac{\sqrt{3}+\sqrt{-5}}{\sqrt{3}-\sqrt{-5}}$.

20. $\dfrac{5+\sqrt{-3}}{2-\sqrt{-3}}$

21. $\dfrac{4+1\sqrt{-2}}{5-2\sqrt{-2}}$

244. Application of radicals. In Chapter XIV, irrational and complex roots of quadratic equations were found. Checking by substitution in such cases was not recommended at that time. It is possible now to check such roots by substitution.

Example. — Solve and check the equation $x^2 + x - 1 = 0$.

Solution. — 1. By the formula, $x = \dfrac{-1 \pm \sqrt{1+4}}{2} = \dfrac{-1 \pm \sqrt{5}}{2}$

2. $\therefore\ r_1 = \dfrac{-1+\sqrt{5}}{2}$; and $r_2 = \dfrac{-1-\sqrt{5}}{2}$.

Check. — Substitute $\dfrac{-1+\sqrt{5}}{2}$ in $x^2 + x - 1 = 0$.

Does $\left(\dfrac{-1+\sqrt{5}}{2}\right)^2 + \left(\dfrac{-1+\sqrt{5}}{2}\right) - 1 = 0$?

Does $\dfrac{1-2\sqrt{5}+5}{4} + \dfrac{-1+\sqrt{5}}{2} - 1 = 0$?

Does $\dfrac{6-2\sqrt{5}-2+2\sqrt{5}-4}{4} = 0$? Yes.

Similarly, $\dfrac{-1-\sqrt{5}}{2}$ will check.

EXERCISE 181

Solve and check by substitution:

1. $x^2 - 2x - 1 = 0$

2. $x^2 + 2x - 2 = 0$

3. $x^2 - 3x + 1 = 0$

4. $x^2 - 2x + 2 = 0$

5. $2x^2 - 2x + 1 = 0$

6. $x^2 + x + 2 = 0$

7. In trigonometry, certain six numbers occur, of which five bear the following relations to the sixth. Calling the numbers $s, c, t, S, C, T,$

a. $c = \sqrt{1 - s^2}$ *c.* $C = \frac{1}{s}$ *e.* $T = \frac{\sqrt{1 - s^2}}{s}$

b. $t = \frac{s}{\sqrt{1 - s^2}}$ *d.* $S = \frac{1}{\sqrt{1 - s^2}}$

If $s = \frac{1}{2}$, find $c, t, C, S,$ and T.

8. In Example 7, find the values of $c, t, S, C,$ and T when $s = \frac{1}{3}$.

IRRATIONAL EQUATIONS

245. **An irrational equation** is an equation in which the variable appears under a radical sign or with a fractional exponent.

The radical sign or the fractional exponent always indicates the principal root in an irrational equation. The sign before the root indicates the operation to be performed on it.

Thus, $-\sqrt{x - 5}$ means "subtract the principal square root of $x - 5$."

Example 1. — Solve the equation $x - 1 - \sqrt{x^2 - 5} = 0$.

Solution. — 1. $x - 1 - \sqrt{x^2 - 5} = 0$

2. $\therefore x - 1 = \sqrt{x^2 - 5}$

3. Squaring both members, $x^2 - 2x + 1 = x^2 - 5$

4. $\therefore -2x = -6$, or $x = 3$.

Check. — Does $3 - 1 - \sqrt{9 - 5} = 0$? Does $2 - \sqrt{4} = 0$? Yes.

NOTE. — When a single radical occurs in an equation, transpose the radical to one side and all other terms to the other side. Then, *if the radical is a square root, square both members of the equation;* if *it is a cube root, cube both members;* etc.

Example 2. — Solve the equation $x - 1 + \sqrt{x^2 - 5} = 0$.

Solution. — 1. $x - 1 + \sqrt{x^2 - 5} = 0$

2. $\therefore +\sqrt{x^2 - 5} = 1 - x$

3. Squaring, $x^2 - 5 = 1 - 2x + x^2$

4. $\therefore 2x = 6$, or $x = 3$.

Check. — Does $3 - 1 + \sqrt{9 - 5} = 0$? Does $2 + \sqrt{4} = 0$? No. *Therefore 3 is not a root of the given equation.*

Recall that a number is sought, in solving an equation, which will satisfy the equation. The equation expresses certain relations which the number must satisfy. *Sometimes the relations imposed are impossible, and in that case no number can be found which will satisfy the equation.* Such is the case in this example. In such cases, the root is only apparently a root, and the equation is only apparently an equation.

What is the explanation of the solution $x = 3$? If the original equation is compared with the equation of Example 1, it is noticed that the only difference is in the sign of the radical; also that in Step 3, after squaring both members in both examples, the resulting equation is the same. In each example, *if the equation of Step 1 has a root*, that number is a root of the equation of Step 3; but, since the equation of Step 3 is the same in each solution, it cannot be asserted in advance whether its root or roots are roots of the equation of Example 1 or of Example 2. When finally the solution $x = 3$ is obtained, the question arises, is 3 a root of the equation in Example 1 or in Example 2? The root $x = 3$ satisfies the equation of Example 1; it does not satisfy the equation of Example 2. It is customary to say that, in Example 2, the *extraneous root 3 is introduced by the method of solution.*

This example makes clear the necessity of checking the solutions of equations.

EXERCISE 182

Solve and check the following equations:

1. $\sqrt{7a+2} - 4 = 0$

2. $4 - 3\sqrt{y} = -4$

3. $\sqrt{7x-3} + 3 = 2x$

4. $\sqrt{13 - 4x} = 2x + 11$

5. $\sqrt[3]{5x^2 - 6x} + 7 = 10$

6. $\sqrt[3]{27x^3 - 27x^2} + 1 = 3x$

7. Solve the equation $\sqrt{x-2} + \sqrt{2x+5} = 3$.

Solution. — 1. $\sqrt{x-2} + \sqrt{2x+5} = 3$

2. $\therefore \sqrt{2x+5} = 3 - \sqrt{x-2}$

3. Squaring, $2x + 5 = 9 - 6\sqrt{x-2} + x - 2.$

(Complete as in Examples 1–6.)

NOTE. — You will get two *apparent roots*, but one will not check. Remember that the radical indicates the *principal root.*

8. $\sqrt{y+5} - \sqrt{y} = 1$

9. $\sqrt{s-3} - 1 = \sqrt{s-10}$

10. $\sqrt{3m-11}+\sqrt{3m+10}=7$

11. $\sqrt{y+7}-\sqrt{y-5}=2$

12. $\sqrt{x+15}-\sqrt{x-24}=13$

13. $\sqrt{c+20}-\sqrt{c-1}=3$ **14.** $\sqrt[3]{4y-40}+14=10$

15. $2\sqrt{3x-2}-3\sqrt{x-2}=2$

16. Solve the equation $\sqrt{y+3}-\sqrt{y+8}=-\sqrt{y}$.

Solution. — 1. Squaring both members:

$$y+3-2\sqrt{(y+3)(y+8)}+y+8=y.$$

(Complete the solution as in Examples 1–6.)

NOTE. — Observe the term $2\sqrt{(y+3)(y+8)}$. It is

$$2\cdot\sqrt{y+3}\cdot\sqrt{y+8}.$$

17. $\sqrt{x+2}-\sqrt{x-2}=\sqrt{2x}$

18. $\sqrt{2t+1}=2\sqrt{t}-\sqrt{t-3}$

19. $\sqrt{x+5}-\sqrt{8-x}=\sqrt{9-2x}$

20. $\sqrt{3-2c}-\sqrt{7+6c}=\sqrt{1+2c}$

21. Solve the equation $\sqrt{z-6}+\sqrt{z}=\dfrac{3}{\sqrt{z-6}}$.

Solution. — 1. $M_{\sqrt{z-6}}$:

$$z-6+\sqrt{z(z-6)}=3.$$

(Complete the solution as in Examples 1–6.)

22. $\sqrt{x+3}+\dfrac{x+3}{\sqrt{x}}=\dfrac{24}{\sqrt{x}}$ **23.** $\sqrt{x}+\sqrt{x-9}=\dfrac{36}{\sqrt{x-9}}$

24. $\dfrac{10x}{\sqrt{10x-9}}+\sqrt{10x+2}=\dfrac{2}{\sqrt{10x-9}}$

25. $\dfrac{\sqrt{t+1}}{\sqrt{t+3}}=\dfrac{\sqrt{t+3}}{\sqrt{t+6}}$

26. $\sqrt{m+3}+\sqrt{3}=\dfrac{6}{\sqrt{m+3}}$

27. $\dfrac{\sqrt{x+2}}{\sqrt{x}}-\dfrac{\sqrt{x}}{\sqrt{x+2}}=\dfrac{2}{3\sqrt{x+2}}$

28. $\dfrac{9+\sqrt{x}}{9-2\sqrt{x}} = \dfrac{11}{5}$ **29.** $\dfrac{\sqrt{3x+1}-\sqrt{8x}}{\sqrt{3x+1}+\sqrt{8x}} = -\dfrac{3}{13}$

30. $\sqrt{27x - \sqrt{10x+5}} = \sqrt{20x+9}$

31. Solve for x: $3\sqrt{ax+a^2} = \sqrt{3ax} + 3a$

32. Solve the equation $t = \pi\sqrt{\dfrac{l}{g}}$:

a. for l *b.* for g

33. Solve the equation $V = \sqrt{2gs}$:

a. for g *b.* for s

34. Solve for s: $\dfrac{3r}{2} = \sqrt{s^2 - \left(\dfrac{s}{2}\right)^2}$

35. Solve for h: $L = \pi r\sqrt{r^2+h^2}$

XXI. LOGARITHMS

246. Logarithms are exponents.

Every positive number can be expressed, exactly or approximately, as a power of 10. The exponent corresponding to a number so expressed is called its **logarithm to the base 10.**

Thus, $10^2 = 100$; therefore 2 is the logarithm of 100 to the base 10. This is written: $\log_{10}100 = 2$, or more briefly $\log 100 = 2$.

Similarly, $\log_{10}35$ is read "logarithm of 35 to the base 10."

247. Multiplication, division, involution, and evolution can be simplified by the use of logarithms. To make this fact clear, the *approximate* values of some powers of 10 will be computed and some examples will be solved.

1. $10^0 = 1$; $10^1 = 10$; $10^2 = 100$; $10^3 = 1000$

2. $10^{.5} = 10^{\frac{1}{2}} = \sqrt{10} = 3.1623$
$10^{1.5} = 10^1 \times 10^{.5} = 10 \times 3.1623 = 31.623$
$10^{2.5} = 10^1 \times 10^{1.5} = 10 \times 31.623 = 316.23$

3. $10^{.25} = (10^{.5})^{\frac{1}{2}} = \sqrt{3.1623} = 1.7782$
$10^{1.25} = 10^1 \times 10^{.25} = 10 \times 1.7782 = 17.782$
$10^{2.25} = 10^1 \times 10^{1.25} = 10 \times 17.782 = 177.82$

4. $10^{.75} = (10^{1.5})^{\frac{1}{2}} = \sqrt{31.623} = 5.6234$
$10^{1.75} = 10 \times 10^{.75} = 10 \times 5.6234 = 56.234$
$10^{2.75} = 10 \times 10^{1.75} = 10 \times 56.234 = 562.34$

$1.0000 = 10^{0.00}$
$1.7782 = 10^{0.25}$
$3.1623 = 10^{0.50}$
$5.6234 = 10^{0.75}$
$10.0000 = 10^{1.00}$
$17.7820 = 10^{1.25}$
$31.6230 = 10^{1.50}$
$56.2340 = 10^{1.75}$
$100.0000 = 10^{2.00}$
$177.8200 = 10^{2.25}$
$316.2300 = 10^{2.50}$
$562.3400 = 10^{2.75}$
$1000.0000 = 10^{3.00}$

NOTE. — Examining the table on the right, observe that $\log_{10} 1000 = 3$; $\log_{10} 177.82 = 2.25$; $\log_{10} 5.6234 = .75$; etc.

Example 1. — Find 3.1623 × 17.782.

Solution. — 1. 3.1623×17.782
2. $= 10^{.50} \times 10^{1.25}$
3. $= 10^{1.75}$
4. $= 56.234$ (in the table)
5. $\therefore\ 3.1623 \times 17.782 = 56.234$

The solution is approximately correct.

Check. —

$$\begin{array}{r} 3.1623 \\ 17.782 \\ \hline 63246 \\ 252984 \\ 221361 \\ 221361 \\ 31623 \\ \hline 56.2320186 \end{array}$$

Example 2. — Find 562.34 ÷ 17.782.

Solution. — 1. $562.34 \div 17.782$
2. $= 10^{2.75} \div 10^{1.25}$
3. $= 10^{2.75-1.25}$
4. $= 10^{1.5}$
5. $= 31.623$ (in the table)
6. $\therefore\ 562.34 \div 17.782 = 31.623$

The solution is approximately correct.

Check. —

$$\begin{array}{r|l} & 31.624 \\ \hline 17.782 & 562.340.000 \\ & 533\ 46 \\ \cline{2-2} & 28\ 880 \\ & 17\ 782 \\ \cline{2-2} & 11\ 098\ 0 \\ & 10\ 669\ 2 \\ \cline{2-2} & 428\ 80 \\ & 355\ 64 \\ \cline{2-2} & 73\ 160 \\ & 71\ 128 \end{array}$$

Example 3. — Find $(5.6234)^2 \times 316.23 \div 177.82$.

Solution. — 1. $(5.6234)^2 \times 316.23 \div 177.82$.
2. $= (10^{.75})^2 \times 10^{2.50} \div 10^{2.25}$
3. $= 10^{1.50+2.50-2.25} = 10^{1.75}$
4. $\therefore\ (5.6234)^2 \times 316.23 \div 177.82 = 56.234$

This solution also may be checked by ordinary computation.

Obviously a more complete list of exponents (logarithms) and ability to use them must be of great advantage, for in each case the solution by exponents is the simpler. The following sections teach how to use logarithms.

248. Logarithms of numbers to the base 10 are called **common logarithms.**

If a number is not an exact power of 10, its logarithm can be given only approximately; a four-place logarithm is one given correct to four decimal places.

Thus the logarithm of 13 is 1.1139; *i.e.* $13 = 10^{1.1139}$, approximately.

The integral part of the logarithm is called the **characteristic** and the decimal part, the **mantissa**.

The characteristic of log 13 is 1 and the mantissa is .1139.

NOTE 1. — The plural of *mantissa* is *mantissæ*.

NOTE 2. — A negative number does not have a logarithm.

249. Finding the characteristic of the logarithm of a number greater than 1.

It is known that $3.53 = 10^{.5478}$, or $\log 3.53 = .5478$.

Multiplying both members of $3.53 = 10^{.5478}$ by 10,

$$35.3 = 10^{.5478} \times 10^1, \text{ or } 10^{1.5478}. \quad \therefore \log 35.3 = 1.5478.$$

Similarly, $353 = 10^{1.5478} \times 10^1$, or $10^{2.5478}$. $\therefore \log 353 = 2.5478$.

The numbers 3.53, 35.3, and 353 have the same *significant figures;* **they differ only in the location of the decimal point. Their logarithms differ only in their characteristics.** These two facts indicate a connection between the location of the decimal point and the characteristic.

3.53 has **one** figure to the left of the decimal point; its logarithm has as characteristic **0**, which is **1 less than 1**.

35.3 has **two** figures to the left of the decimal point; its logarithm has as characteristic 1, which is **1 less than 2**.

353 has **three** figures to the left of the decimal point; its logarithm has as characteristic 2, which is **1 less than 3**.

Rule. — The characteristic of the common logarithm of a number greater than 1 is one less than the number of significant figures to the left of the decimal point.

Thus, the characteristic of log 357.83 is 2; of log 70390.5 is 4.

EXERCISE 183

What is the characteristic of the logarithm of:

1. 59
2. 540
3. 4000
4. 8
5. 72,860
6. 11.2
7. 367.25
8. 50900.4
9. 5.08
10. 30,027.8
11. 21.678
12. 100.005
13. 984.26
14. 87,600.
15. 2.1932
16. 1,000,000

Tell the number of significant figures preceding the decimal point when the characteristic of the logarithm is:

17. 5 **18.** 3 **19.** 0 **20.** 1 **21.** 4 **22.** 2

250. Finding the characteristic of the logarithm of a number less than 1. Dividing both members of $3.53 = 10^{.5478}$ (§ 249) by 10,

$.353 = 10^{.5478} \div 10^1 = 10^{.5478-1}$. $\therefore \log .353 = .5478 - 1$.

Dividing both members of $.353 = 10^{.5478-1}$, by 10,

$.0353 = 10^{.5478-1} \div 10^1 = 10^{.5478-2}$. $\therefore \log .0353 = .5478 - 2$.

Similarly, $.00353 = 10^{.5478-3}$. $\therefore \log .00353 = .5478 - 3$.

Between the decimal point and the first significant figure of:

.353 there are no zeros; the characteristic of log .353 is -1.

.0353 there is one zero; the characteristic of log .0353 is -2.

.00353 there are two zeros; the characteristic of log .00353 is -3.

Rule. — The characteristic of the common logarithm of a (positive) number less than 1 is negative; numerically it is one more than the number of zeros between the decimal point and the first significant figure.

Thus, the characteristic of log .0045 is -3; of log .00027, is -4.

EXERCISE 184

What is the characteristic of the logarithm of:

1. .5
2. .004
3. .00082
4. .07
5. .01003
6. .00093
7. .64325
8. .04216
9. .00205
10. .1007
11. .00008
12. .24038

Tell the number of zeros preceding the first significant figure when the characteristic of the logarithm is:

13. -2
14. -8
15. -4
16. -10
17. -1
18. -6
19. -5
20. -7
21. -3
22. -9

251. Another method of writing a negative characteristic. In § 250 log .353 = .5478 − 1. Actually, therefore, log .353 is − .4522, a negative number. However, the positive mantissa and the negative characteristics are retained.

.5478 − 1 may be written: 9.5478 − 10. Numerically the two expressions have equal value. Note that 9 − 10 = − 1.

The process in general is to decide upon the characteristic by the rule in § 250; then, if it is − 1, write it 9 − 10; if − 2, write it 8 − 10; etc.

Thus, log .02 is .3010 − 2, or 8.3010 − 10.

NOTE. — The negative characteristic is often written thus: log 02 = $\bar{2}$.3010; again, log .353 = $\bar{1}$.5478. The minus sign is written over the characteristic to indicate that it alone is negative, the mantissa being positive.

EXERCISE 185

1–12. Tell how each of the characteristics of the examples of Exercise 184 should be written.

252. Mantissa of a logarithm. From §§ 249 and 250:

log 3.53 = .5478; log .353 = 9.5478 − 10;
log 35.3 = 1.5478; log .00353 = 7.5478 − 10.

The numbers 3.53, 35.3, .353, and .00353 have the same significant figures. Their common logarithms have the same mantissæ. This is an example of the

Rule. — The common logarithms of all numbers having the same significant figures have the same mantissæ.

Thus, the logarithms of 2506, 2.506, 250.6, etc., all have the same mantissæ.

253. A table of logarithms consists of the mantissæ of the logarithms of certain numbers. The characteristics of the logarithms are determined by the rules given in §§ 249 and 250. The table given on pages 370 and 371 gives the mantissæ of all integers from 100 to 999 inclusive, calculated to four decimal places. The decimal point is omitted. Such a table is called a *four-place table.* While a five or six place table would be more accurate, this table is sufficiently accurate for all ordinary purposes.

254. Finding the logarithm of a number of three significant figures.

Example 1. — Find the logarithm of 16.8.

Solution. — 1. In the column headed "No." find 16. On the horizontal line opposite 16, pass over to the column headed by the figure 8. The mantissa 2253 found there, is the required mantissa.

2. The characteristic is 1, by the rule in § 249.

3. $\therefore$ log 16.8 is 1.2253.

Rule. — To find the logarithm of a number of three figures:

1. Look in the column headed "No." for the first two figures of the given number. The mantissa will be found on the horizontal line opposite these two figures and in the column headed by the third figure of the given number.

2. Prefix the characteristic according to §§ 249 and 250.

Example 2. — Find log .304.

Solution. — 1. Opposite 30 in the column headed by 4 is the mantissa .4829. The characteristic is -1 or $9 - 10$. (§§ 250 and 251.)

2. $\therefore \log .304 = 9.4829 - 10.$

NOTE. — The logarithm of a number of one or two significant figures may be found by using the column headed 0. Thus the mantissa of log 8.3 is the same as the mantissa of log 8.30; of log 9, the same as of log 900.

EXERCISE 186

Find the logarithm of:

1. 365
2. 571
3. 847
4. 902
5. 200
6. 64
7. 9
8. 5.2
9. 43.6
10. 720
11. .841
12. .0628
13. .00175
14. 7680
15. 25900
16. .000834
17. .07
18. 3.14
19. 40.8
20. .16

255. Finding the logarithm of a number of more than three significant figures.

Example 1. — Find log 327.5.

Solution. — 1. From the table

$$\left.\begin{aligned} \log 327 &= 2.5145 \\ \log 327.5 &= \quad ? \\ \log 328 &= 2.5159 \end{aligned}\right\} \begin{gathered}\text{Difference} \\ = .0014.\end{gathered}$$

2. Since 327.5 is between 327 and 328, its logarithm must be between their logarithms. An increase of one unit in the number (from 327 to 328) produces an increase of .0014 in the mantissa. It is *assumed* therefore that an increase of .5 in the number (from 327 to 327.5) produces an increase of .5, of .0014, or of .0007 in the mantissa.

3.

$$\begin{aligned} \therefore \log 327.5 &= 2.5145 + .5 \times .0014 \\ &= 2.5145 + .0007 = 2.5152. \end{aligned}$$

This result is obtained in practice as follows. The difference between any mantissa and the next higher mantissa as written in the table (neglecting the decimal point) is called the *tabular difference.* The tabular difference for this example is 14(5159–5145). .5 of the tabular difference is 7. Adding this to 5145 gives 5152, the required mantissa of log 327.5.

Similarly to find log 327.25, the tabular difference is 14. $.25 \times 14 = 3.5$. Hence the mantissa of log 327.25 is $5145 + 3.5$ or 5148.5. $\therefore \log 327.25 = 2.5149$.

NOTE 1. — The process of determining a mantissa which is between two mantissæ of the table is called *interpolation.*

NOTE 2. — The assumption made in Step 2 is not warranted by the facts. Nevertheless, for ordinary purposes, the results obtained in this manner are sufficiently correct.

NOTE 3. — When interpolating, it is customary to cut down all decimals so that the mantissa will again be a four-place decimal. Thus 3.5 is called 4. 3.4 would be called 3.

Rule. — To find the logarithm of a number of more than three significant figures:

1. Find the mantissa for the first three figures, and the tabular difference for that mantissa. (See Step 3 above.)

2. Multiply the tabular difference by the remaining figures of the given number, preceded by a decimal point. (See Note 3 above.)

No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
No.	0	1	2	3	4	5	6	7	8	9

No.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
No.	0	1	2	3	4	5	6	7	8	9

3. Add the result of Step 2 to the mantissa obtained in Step 1.

4. Prefix the proper characteristic, determined by §§ 249 and 250.

Example 2. — Find log 34.652.

Solution. — 1. Mantissa of log 346 = 5391
Mantissa of log 347 = 5403
2. Tabular difference = 12. $.52 \times 12 = 6.24$, or 6
3. $\therefore$ Mantissa for log 34652 = 5391 + 6, or 5397
4. $\therefore$ log 34.652 = 1.5397.

Example 3. — Find log .021508.

Solution. — 1. Mantissa of log 215 = 3324
Mantissa of log 216 = 3345
2. Tabular difference = 21. $.08 \times 21 = 1.68$, or 2
3. $\therefore$ mantissa of log 21508 = 3324 + 2, or 3326
4. $\therefore$ log .021508 = .3326 − 2, or 8.3326 − 10.

EXERCISE 187

What is the tabular difference *when the mantissa* is:

1. 2380
2. 3243
3. 5752
4. 7427
5. 8949
6. 4771
7. 9595
8. 1461
9. 5237
10. 7774

Find the logarithm of:

11. 342.5
12. 252.1
13. 865.2
14. 764.4
15. 438.3
16. 501.6
17. 28.25
18. 1.158
19. 7.631
20. .5842
21. .04873
22. 328.15
23. 453.24
24. 86.431
25. 3.7275
26. 1.0673
27. .052425
28. 3.1416
29. 6324.5
30. .008004

256. Finding the number corresponding to a given logarithm.

Example 1. — Find the number whose logarithm is 1.6571.

Solution. — 1. Find the mantissa 6571 in the table.

2. In the column headed "No." on the line with 6571 is 45. These are the first two figures of the number. At the head of the column containing 6571 is 4, the third figure of the number. Hence the number sought has the figures 454.

3. The characteristic being 1, the number must have two figures to the left of the decimal point. (§ 249)

4. $\therefore$ the number is 45.4.

Rule. — To find the number corresponding to a given logarithm when the mantissa appears in the table:

1. Find the three figures corresponding to this mantissa, as in Example 1.

2. Place the decimal point according to the rules in §§ 249 and 250.

EXERCISE 188

Find the number whose logarithm is:

1. 3.4786
2. 1.7300
3. .9325
4. 2.2095
5. 3.8615
6. 0.6284
7. 1.3962
8. 2.5105
9. $9.2405 - 10$
10. $8.7966 - 10$
11. $7.9868 - 10$
12. $9.4728 - 10$
13. 5.9149
14. 3.6776
15. 4.8899
16. 1.3010

Example 2. — Find the number whose logarithm is 1.3934.

Solution. — 1. The mantissa 3934 does not appear in the table. The next less mantissa is 3927, and the next greater is 3945. The corresponding numbers are 247 and 248. That is:

$$\left.\begin{array}{l}\left.\begin{array}{l}\text{mantissa of }\log 247 = 3927\\ \text{mantissa of }\log x \;\;\;= 3934\end{array}\right\}\begin{array}{l}\text{Diff.}\\ = 7.\end{array}\\ \text{mantissa of }\log 248 = 3945\end{array}\right\}\begin{array}{l}\text{Tabular}\\ \text{difference}\\ = 18.\end{array}$$

2. Since an increase of 18 in the mantissa produces an increase of 1 in the number, it is *assumed* that an increase of 7 in the mantissa must produce an increase of $\frac{7}{18}$ or .38 in the number. Hence the number has the figures 247.38.

3. Since the characteristic is 1, the number must be 24.738.

Rule. — To find the number corresponding to a given logarithm when the mantissa does not appear in the table:

1. Find in the table the next less mantissa. Find the corresponding number of three figures, and the tabular difference.

2. Subtract the next less mantissa from the given mantissa, and divide the remainder by the tabular difference to two decimal places.

3. Annex the quotient to the number of three figures obtained in Step 1.

4. Place the decimal point according to the rules in §§ 249 and 250.

EXERCISE 189

Find the number whose logarithm is:

1. 2.7408
2. 1.6678
3. .4188
4. 3.8983
5. 2.9417
6. 9.5969−10
7. 8.3429−10
8. 3.8497
9. 7.3288−10
10. .1195
11. 2.2930
12. 1.9665
13. 3.8598
14. 9.7606−10
15. 4.3346
16. 8.9194−10
17. 7.7978−10
18. 3.2306
19. 1.8817
20. 2.9986

PROPERTIES OF LOGARITHMS

257. The preceding discussion relates entirely to the common system of logarithms (§ 248). Certain properties of logarithms to any base will be considered now.

Note. — The base may be any positive number different from 1.

258. Just as $\log_{10} 3.053 = .4847$ means that $10^{.4847} = 3.053$, so $\log_a N = x$ means that $N = a^x$.
$\log_a N$ is read "the logarithm of N to the base a."

259. Logarithm of a product.

$$\left.\begin{array}{ll}\text{Assume that} & a^x = M\\ \text{and} & a^y = N\end{array}\right\};\ \text{then}\ \left\{\begin{array}{l} x = \log_a M,\\ y = \log_a N.\end{array}\right.$$

Also $a^x \cdot a^y = MN$, or $a^{x+y} = MN$. $\therefore \log_a MN = x + y$. (§ 258)

$$\text{Therefore } \log_a MN = \log_a M + \log_a N.$$

Rule. — In any system, the logarithm of a product is equal to the sum of the logarithms of its factors.

Example. — Given $\log 2 = .3010$, and $\log 3 = .4771$, find $\log 72$.

Solution. —1. $\log 72 = \log 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$

2. $= \log 2 + \log 2 + \log 2 + \log 3 + \log 3$

3. $\therefore \log 72 = 3 \log 2 + 2 \log 3$, or $3(.3010) + 2(.4771)$

$= .9030 + .9542$, or 1.8572.

EXERCISE 190

Given $\log 2 = .3010$, $\log 3 = .4771$, and $\log 5 = .6990$. Find the following logarithms as above; check the solutions by finding the same logarithms in the table:

1. log 15
2. log 30
3. log 60
4. log 45
5. log 36
6. log 150
7. log 144
8. log 256
9. log 360
10. log 240
11. log 432
12. log 375

13. Find by logarithms the value of $35.2 \times 2.35 \times 6.43$.

Solution. — 1. Let $v = 35.2 \times 2.35 \times 6.43$
2. $\therefore \log v = \log 35.2 + \log 2.35 + \log 6.43$
3. $\therefore \log v = 2.7258$
4. $\therefore \quad v = 531.87$ (§ 256).

$$\begin{aligned} \log 35.2 &= 1.5465 \\ \log 2.35 &= 0.3711 \\ \log 6.43 &= \underline{0.8082} \\ & \ 2.7258 \end{aligned}$$

Find by logarithms the value of:

14. 42.5×26.8
15. 3.89×72.6
16. $535 \times .621$
17. 239.5×38.4
18. 871.2×45
19. 1.414×360
20. 7.026×8059
21. 432.4×1.658
22. 93.62×768.75

23. Find by logarithms the value of $.0631 \times 7.208 \times .51272$.

Solution. — 1. $\log v = \log .0631 + \log 7.208 + \log .51272$

2.
$$\begin{aligned} \log .0631 &= 8.8000 - 10 \\ \log 7.208 &= 0.8578 \\ \log .51272 &= \underline{9.7099 - 10} \\ & \ 19.3677 - 20 = 9.3677 - 10 \end{aligned}$$

3. $\therefore \log v = 9.3677 - 10 \qquad \therefore v = .2332$.

NOTE. — If the sum of the logarithms is a negative number, the result should be written so that the negative part of the characteristic is -10. In this example, the characteristic $19 - 20$ is changed to $9 - 10$.

Find by logarithms the value of:

24. $.342 \times 2.15$

25. $.6546 \times .0368$

26. $.7932 \times 8.421$

27. $76.85 \times .00435$

28. $.0169 \times .347 \times .846$

29. $.081 \times .3806 \times 563$

260. Logarithm of a quotient.

Assume that $a^x = M$
and $a^y = N$ $\Big\}$; then $\begin{cases} x = \log_a M, \\ y = \log_a N. \end{cases}$

Also, $a^x \div a^y = M \div N$, or $a^{x-y} = M \div N$.

$$\therefore \log_a (M \div N) = x - y.$$

Therefore, $\log_a (M \div N) = \log_a M - \log_a N$.

Rule. — In any system, the logarithm of the quotient of two numbers is equal to the logarithm of the dividend minus the logarithm of the divisor.

Example 1. — Given $\log 2 = .3010$ and $\log 3 = .4771$, find $\log \frac{3}{2}$.

Solution. — 1. $\log \frac{3}{2} = \log 3 - \log 2 = .4771 - .3010 = .1761$.

Example 2. — Find $\log \frac{8}{9}$.

Solution. — 1. $\log \frac{8}{9} = \log \frac{2 \cdot 2 \cdot 2}{3 \cdot 3}$

2. $= (\log 2 + \log 2 + \log 2) - (\log 3 + \log 3)$
3. $= 3(.3010) - 2(.4771) = .9030 - .9542$
4. $\therefore \log \frac{8}{9} = 9.9488 - 10$.

$$\begin{array}{rl} .9030 = & 10.9030 - 10 \\ .9542 = & \underline{.9542} \\ & 9.9488 - 10 \end{array}$$

Note 1. — To subtract a logarithm from a smaller logarithm, or to subtract a negative logarithm from a positive one, increase the characteristic of the minuend by 10, writing -10 after the mantissa to compensate. Thus, in this example, .9542 is greater than .9030; therefore, .9030 is written $10.9030 - 10$, after which the subtraction is performed.

Note 2. — To find the logarithm of a fraction, add the logarithms of the factors of the numerator, and from the result subtract the sum of the logarithms of the factors of the denominator.

EXERCISE 191

Given $\log 2 = .3010$, $\log 3 = .4771$, and $\log 5 = .6990$. Find:

1. $\log \frac{5}{3}$ **3.** $\log \frac{20}{9}$ **5.** $\log \frac{27}{25}$ **7.** $\log \frac{6}{5}$

2. $\log \frac{10}{3}$ **4.** $\log \frac{12}{5}$ **6.** $\log \frac{45}{8}$ **8.** $\log \frac{27}{64}$

Find by logarithms the value of:

9. $335 \div 56$ **12.** $230.4 \div 125$ **15.** $3305 \div 1.414$

10. $483 \div 71$ **13.** $739.75 \div 1.73$ **16.** $8.964 \div 45.25$

11. $688 \div 49$ **14.** $400.36 \div 2.53$ **17.** $6.429 \div 56.84$

18. $\frac{4.16 \times 32}{485}$ **22.** $\frac{3.25 \times .0063}{.007}$

19. $\frac{35.2 \times 1.52}{53.87}$ **23.** $\frac{527.8 \times .069}{2.449}$

20. $\frac{43.57 \times .069}{3.14}$ **24.** $\frac{4.501 \times 837 \times .0239}{256 \times 324}$

21. $\frac{14.07 \times 347}{18}$ **25.** $\frac{302 \times 1.414 \times .0293}{450 \times .3645}$

261. **The logarithm of a power of a number.**

Assume that $a^x = M$; or $x = \log_a M$.

Then, $(a^x)^p = M^p$, or $a^{px} = M^p$. $\therefore \log_a M^p = px$

Therefore, $\log_a M^p = p \log_a M$.

Rule. — In any system, the logarithm of any power of a number is equal to the logarithm of the number multiplied by the exponent indicating the power.

Example 1. — Given $\log 7 = .8451$, find $\log 7^5$.

Solution. — $\log 7^5 = 5 \log 7 = 5 \times .8451 = 4.2255$

Example 2. — Find by logarithms 1.04^{10}.

Solution. — 1. $\log 1.04^{10} = 10 \log 1.04 = 10 \times .0170 = .1700$

2. The number whose logarithm is .1700 is 1.479. (§ 256)

3. $\therefore 1.04^{10} = 1.479$.

Example 3. — Find by logarithms $\sqrt[3]{365}$.

Solution. — 1. $\log \sqrt[3]{365} = \log 365^{\frac{1}{3}} = \frac{1}{3} \log 365$

2. $\therefore \log \sqrt[3]{365} = \frac{1}{3} \times 2.5623 = 0.8541$

3. The number whose logarithm is 0.8541 is 7.146. (§ 256)

4. $\therefore \sqrt[3]{365} = 7.146.$

When finding a cube root, the logarithm of the radicand is divided by 3; when finding a square root, the logarithm of the radicand is divided by 2. This suggests the

Rule. — In any system, the logarithm of a root of a number is the logarithm of the radicand divided by the index of the root.

Example 4. — Find by logarithms $\sqrt[4]{.0359}$.

Solution. — 1. $\log \sqrt[4]{.0359} = \frac{1}{4} \log .0359 = \frac{1}{4}(8.5551 - 10)$

2. $\therefore \log \sqrt[4]{.0359} = \frac{1}{4}(38.5551 - 40)$ (See note below.)

3. $\therefore \log \sqrt[4]{.0359} = 9.6387 - 10$

4. The number whose logarithm is $9.6387 - 10$ is .4352. (§ 256)

5. $\therefore \sqrt[4]{.0359} = .4352.$

NOTE. — To divide a negative logarithm, write it in such form that the negative part of the characteristic may be divided exactly by the divisor, and give -10 as quotient.

Thus, $8.5551 - 10$ is changed to $38.5551 - 40$ since the divisor is 4. If the divisor were 3, it would be changed to $28.5551 - 30$.

EXERCISE 192

Given $\log 2 = .3010$, $\log 3 = .4771$, and $\log 5 = .6990$. Find:

1. $\log 3^7$
2. $\log 3^5$
3. $\log 5^4$
4. $\log 15^5$
5. $\log (30)^{\frac{1}{3}}$
6. $\log \sqrt[4]{5}$
7. $\log \sqrt[3]{10}$
8. $\log \sqrt[6]{15}$

Find by logarithms the value of:

9. 323^2
10. 4.025^2
11. $\sqrt{418.5}$
12. $\sqrt[3]{784}$
13. $\sqrt[4]{92.04}$
14. 8.975^2
15. $\frac{4}{3} \times 3.1416 \times 6.5^3$
16. 3.1416×19^2

17. $\sqrt{\dfrac{260}{894.5}}$

18. $\sqrt{32.2 \times 20.5 \times 12.8}$

19. $\sqrt[3]{6} \times \sqrt[4]{7}$

20. $\left(\dfrac{3967}{7540}\right)^2$

21. The volume of a right circular cylinder is given by the formula $V = \pi R^2 H$. Find V by logarithms:
a. if $R = 12.4$ and $H = 30.3$. *b.* if $R = 7.6$ and $H = 28.5$.

22. The distance covered by a freely falling body is given by the formula $s = \frac{1}{2}gt^2$. Find s if $g = 32.16$ feet and $t = 4.5$ seconds.

23. The volume of a cone is given by the formula $V = \frac{1}{3}\pi R^2 H$. Find V if $R = 9.5$ and $H = 22$.

24. The volume of a sphere is given by the formula $V = \frac{4}{3}\pi R^3$. Find V: *a.* if $R = 5.4$. *b.* if $R = 10.5$.

25. The simple interest formula is $I = PRT$. Find I:
a. if $P = \$845$; $R = 4\%$; and $T = 5.5$ yr.
b. if $P = \$1850$; $R = 3.5\%$; and $T = 7$ yr. 9 mo.

26. The amount to which P dollars will accumulate at $R\%$ compounded annually for n years is given by the formula $A = P\left(1 + \dfrac{R}{100}\right)^n$. Find A:
a. if $P = \$450$; $R = 4\%$; and $n = 8$ yr.
b. if $P = \$125$; $R = 3.5\%$; and $n = 20$ yr.

27. The area of an equilateral triangle whose side is s inches long is given by the formula $A = \dfrac{s^2}{4}\sqrt{3}$. Find A when $s = 12$.

28. Using the formula $F = \dfrac{mv^2}{gr}$, find F when $m = 150$, $g = 32$, $v = 25$, and $r = 5$.

29. Using the formula $Z = 2\pi rh$, find Z when $\pi = \frac{22}{7}$, $r = 11.2$, and $h = 3$.

30. Using the formula $S = \frac{1}{2}l \cdot \pi(r + s)$, find S when $l = 9.7$, $r = 3.6$, and $s = 5.7$.

XXII. PROGRESSIONS

ARITHMETIC PROGRESSION

262. An **arithmetic progression** (A. P.) is a sequence of numbers, called **terms**, each of which after the first is derived from the preceding by adding to it a fixed number, called the **common difference**.

Thus, 1, 3, 5, 7, ···· is an A. P. Each term after the first is derived from the preceding by adding 2. The next two terms are 9 and 11. 2 is the common difference.

Again, 9, 6, 3, ···· is an A. P. The common difference is -3. The next two terms are 0 and -3.

NOTE. — The common difference is found by subtracting any term from the one following it.

EXERCISE 193

Determine which of the following are arithmetic progressions; determine the common difference and the next two terms of the arithmetic progressions:

1. 5, 8, 11, 14, ····

2. 3, 8, 13, 18, ····

3. 1, 5, 8, 13, ····

4. 32, 26, 20, 14, ····

5. $1\frac{2}{3}, 2\frac{1}{3}, 3, 3\frac{2}{3}$, ····

6. $6r, 8.5r, 11r$, ····

7. $3a, .5a, -2a$, ····

8. 1.05, 1.10, 1.15

9. $c + d, 2c + d, 3c + d$

10. $6m + 8n, 4m + 9n, 2m + 10n$, ····

Write the first five terms of the A. P.	**11**	**12**	**13**	**14**	**15**
in which the first term is . . .	13	22	3.5	y	a
and the common difference is . .	6	-7	4.5	-5	d

16. Find the 15th term of the A. P. whose first term is 5, and common difference 7.

263. The nth term of an arithmetic progression. It is possible to determine a particular term of an arithmetic progression without finding all of the preceding terms.

Given — the first term a, the common difference d, and the number of the term, n, of an arithmetic progression.

Find — the nth term, l.

Solution. — 1. The progression is

TERM 1	TERM 2	TERM 3	TERM 4	TERM 10	TERM n
a	$a + d$	$a + 2d$	$a + 3d$	····?	····?

2. The coefficient of d in each term is 1 less than the number of the term.

3. What is the 10th term? the 18th? the 35th?

4. Similarly, the coefficient of d in the nth term is $(n - 1)$.

5. $\therefore \quad \boxed{l = a + (n - 1)d}$

Example. — Find the 10th term of 8, 5, 2, ····.

Solution. — 1. This is an A. P. in which $a = 8$, $d = -3$, $n = 10$, and $l = ?$

2. The formula is $l = a + (n - 1)d$.

3. $\therefore l = 8 + 9(-3) = 8 - 27$, or -19.

EXERCISE 194

Find:

1. The 10th term of 4, 10, 16, ····; also the 18th.
2. The 12th term of 17, 14, 11, ····; also the 26th.
3. The 16th term of -3, -8, -13, ····; also the 36th.
4. The 21st term of 2, 2.05, 2.10, ····; also the 46th.
5. The 11th term of 1, $1\frac{1}{2}$, 2, ····; also the 52d.
6. What term of the progression 1.05, 1.10, 1.15, ···· is 2.25?

Solution. — 1. This is an A. P. in which $a = 1.05$, $d = .05$, $n = ?$ and $l = 2.25$. The formula is $l = a + (n - 1)d$.

2. $\therefore 2.25 = 1.05 + (n - 1) \cdot .05$.

(Complete the solution.)

$l = -3 + ((15)(-5))$

-78

7. What term of the progression 7, 4, 1, ···· is -80?

8. What term of the progression -51, -43, -35, ···· is 205?

9. What term of the progression $\frac{1}{6}$, $\frac{4}{3}$, $\frac{5}{2}$, ···· is $23\frac{1}{2}$?

10. What term of the progression 1.07, 1.14, 1.21, ···· is 3.17?

Find the common difference:

11. If the first term of an A. P. is 6 and the 27th term is 188.

12. If the first term is 8 and the 21st term is 108.

13. If the first term is -25 and the 23d term is -14.

14. If the first term is 430 and the 41st term is 110.

15. Find the 12th term of the A. P. whose first term is 10 and whose 19th term is 100.

16. A man is paying for a house on the installment plan. His payments during the first three months are $20.00, $20.10, and $20.20. What will his 20th and 30th payments be?

17. The first three payments on a washing machine are $8.00, $8.04, and $8.08. What will the 12th payment be?

18. In a Christmas savings fund, the payments to be made for 50 weeks are 5¢, 10¢, 15¢, etc. What will the 40th payment be? the 50th?

19. Another savings fund plan calls for payments for 50 weeks as follows: $5.00, $4.90, $4.80, etc. What will the 25th payment be? the 50th?

264. The terms of an arithmetic progression between any two other terms are called the **arithmetic means** of these two terms.

Thus, the *three arithmetic means* of 2 and 14 are 5, 8, 11, since 2, 5, 8, 11, 14 form an arithmetic progression.

A single arithmetic mean of two numbers is particularly important. It is called **the arithmetic mean** of the numbers.

When two numbers are given, any specified number of arithmetic means can be inserted between them.

Example. — Insert five arithmetic means between 13 and -11.

Solution. — 1. There will be an arithmetic progression of 7 terms, in which $a = 13$, $l = -11$, and $n = 7$. Find d.

2. $l = a + (n - 1)d$. $\therefore -11 = 13 + 6d$, or $d = -4$.

3. The progression is: 13, 9, 5, 1, -3, -7, -11.

Check. — This is an A. P. with five terms between 13 and -11.

EXERCISE 195

1. Insert three arithmetic means between 1 and 17.

2. Insert four arithmetic means between -14 and 16.

3. Insert seven arithmetic means between 8 and 20.

4. Insert five arithmetic means between $1\frac{1}{2}$ and 6.

5. Insert four arithmetic means between $-\frac{5}{3}$ and -5.

6. Find the arithmetic mean of 5 and 13.

7. Find the arithmetic mean of $(y - 4)$ and $(y + 4)$.

8. Find the arithmetic mean of $\sqrt{3}$ and $\sqrt{27}$.

9. Find the arithmetic mean of a and b. From the result, make a rule for finding the arithmetic mean of any two numbers.

NOTE. — *The* arithmetic mean of two numbers is also called *their average.*

10. *a.* Find the arithmetic mean of 11 and 19.

b. Check your results by finding this arithmetic mean by use of the formula derived in Example 9.

11. Find the common difference if 3 arithmetic means are inserted between g and h.

12. Find the common difference if r arithmetic means are inserted between 5 and 45.

13. Find the common difference if r arithmetic means are inserted between s and t.

265. **The sum of the first n terms of an arithmetic progression** can be found without writing down all the terms.

Given — the first term, a, the number of terms, n, and the nth term, l.

Find — the sum of these n terms, S.

Solution. — 1. $S = a + (a + d) + (a + 2d) \cdots + (l - 2d) + (l - d) + l.$

2. Writing the terms in reverse order,

$$S = l + (l - d) + (l - 2d) \cdots + (a + 2d) + (a + d) + a.$$

3. Adding the equations in Steps 1 and 2,

$$2S = (a + l) + (a + l) + (a + l) \cdots + (a + l) + (a + l) + (a + l).$$

4. Since there were n terms in equations 1 and 2, and since a sum $(a + l)$ results in Step 3 from each term of equation 1,

$$2S = n(a + l).$$

5. $$\therefore \boxed{S = \frac{n}{2}(a + l)}$$

6. By § 263, $l = a + (n - 1)d$. Substituting this value of l in Step 5,

$$S = \frac{n}{2}[a + \{a + (n - 1)d\}].$$

7. $$\therefore \boxed{S = \frac{n}{2}\{2a + (n - 1)d\}}$$

NOTE. — The formulæ in Steps 5 and 7 should be memorized.

Example. — Find the sum of the first 12 terms of the progression 8, 5, 2, ⋯.

Solution. — 1. $a = 8;\quad d = -3;\quad n = 12;\quad S = ?$

2. $S = \frac{n}{2}\{2a + (n - 1)d\}. \quad \therefore S = 6\{16 + 11 \cdot (-3)\}.$

3. $\therefore S = 6(-17) = -102.$

EXERCISE 196

Find the sum of:

1. 14 terms of 4, 9, 14.

2. 18 terms of 6, 1, − 4.

3. 15 terms of − 80, − 71, − 62.

4. 12 terms of \$1.08, \$1.16, \$1.24.

Find the sum of the terms of an arithmetic progression if:

5. The number is 16, the first is 3, and the last is 48.

6. The number is 39, the first is 0, and the last is 50.

7. The number is 19, the first is 24, and the last is -48.

8. The number is 6, the first is $-\frac{2}{7}$, and the last is $\frac{5}{14}$.

9. Find the sum of the even numbers from 2 to 50.

10. Find the sum of the numbers 3, 6, 9, $\cdots\cdot$ 99.

11. Find the sum of the odd numbers from 1 to 199.

12. Find the sum of all the even integers beginning with 2 and ending with 400.

13. A pile of fence posts has 40 in the first layer, 39 in the second, 38 in the third. There are 20 layers. How many posts are there in the pile?

14. In a certain school system, a teacher is paid $950 for her first year's work, and is given an increase of $50 per year each year thereafter. What will be the teacher's total income during ten years of service?

15. On a debt of $4000, a man was paying 7% interest. At the end of each year, he plans to pay $200 on the principal, and also the interest on the balance of the debt during that year.

a. Write down his payments to be made at the end of years one, two, and three.

b. How many years will it take him to pay off the debt?

c. How much will he have paid by the time he has freed himself of the debt?

16. On a large apartment building there is a debt of $75,000, with interest at 6%, payable at the end of each year. Each year, $5000 of the debt is paid off, thus decreasing both the debt and the interest to be paid each year.

a. What is the payment due at the end of years one, two, and three?

b. How much will have been paid for interest by the time the debt is paid off?

17. It has been learned that if a marble, placed in a groove on an *inclined plane*, passes over a distance D in one second, then in the second second it will pass over the distance $3\,D$, in the third, over the distance $5\,D$, etc. Over what distance will it pass in the 10th second? in the tth second?

18. Through what total distance does the marble in Example 17 pass in 5 seconds? in 10 seconds? in t seconds?

19. Experiment has shown that an object will fall during successive seconds the following distances:

1st second, 16.08 ft.; 3d second, 80.40 ft.;
2d second, 48.24 ft.; 4th second, 112.56 ft.

Find the distance through which the object will fall during the 7th second; the tth second.

20. Find the total distance through which the object in Example 19 falls in 5 seconds; in t seconds.

21. Substitute g for 32.16 in the final result of Example 20 and simplify the result.

SUPPLEMENTARY TOPIC

266. In an arithmetic progression, there are five elements. a, d, l, n, S. Two independent formulæ connect these elements, the formula for the sum and the formula for the term l. Hence, if any three of the elements are known, the other two may be found.

NOTE. — Remember that the formula for the sum is given in two ways.

Example 1. — Given $a = -\frac{5}{3}$, $n = 20$, $S = -\frac{5}{3}$; find d and l.

Solution. —

1. $S = \frac{n}{2}(a + l)$. $\therefore\ -\frac{5}{3} = 10\left(-\frac{5}{3} + l\right)$; whence $l = \frac{3}{2}$.

2. $l = a + (n - 1)d$. $\therefore\ \frac{3}{2} = -\frac{5}{3} + (19) \cdot d$; whence $d = \frac{1}{6}$.

Example 2. — Given $a = 7$, $d = 4$, $S = 403$; find n and l.

Solution. —

1. $S = \frac{n}{2}\{2a + (n-1)d\}$. $\therefore 403 = \frac{n}{2}\{14 + (n-1)\cdot 4\}$.

2. $\therefore 806 = n\{4n + 10\}$; $4n^2 + 10n - 806 = 0$;

$$2n^2 + 5n - 403 = 0.$$

3. $$\therefore n = \frac{-5 \pm \sqrt{25 + 3224}}{4} = \frac{-5 \pm \sqrt{3249}}{4}$$

$$= \frac{-5 \pm 57}{4}.$$

$$\therefore n = -\frac{62}{4}, \text{ or } +13.$$

Since n, the number of terms, must be an integer, n must be 13, and not $-\frac{62}{4}$.

4. $$\therefore l = 7 + 12 \cdot 4 = 55.$$

NOTE. — A negative or a fractional value on n must be rejected, together with all other values depending upon it.

Example 3. — The sixth term of an arithmetic progression is 10 and the 16th term is 40. Find the 10th term.

Solution. — 1. By the formula $l = a + (n-1)d$:

$$a + 5d = 10.$$
$$a + 15d = 40.$$

2. Solving the system of equations in Step 1, $d = 3$ and $a = -5$.

3. The 10th term: $l = -5 + 9 \cdot 3 = -5 + 27 = 22$.

EXERCISE 197

1. Given $l = 57$, $d = 4$, $n = 17$; find a and S.
2. Given $a = -7$, $l = 123$, $n = 27$; find d and S.
3. Given $a = \frac{3}{4}$, $l = \frac{15}{2}$, $S = \frac{165}{4}$; find n and d.
4. Given $a = \frac{3}{2}$, $l = -\frac{3}{2}$, $d = -\frac{1}{9}$; find n and S.
5. Given $a = \frac{2}{3}$, $n = 16$, $S = \frac{232}{3}$; find l and d.
6. Given $n = 19$, $S = -\frac{57}{2}$, $d = \frac{1}{2}$; find a and l.
7. Given $a = -\frac{7}{2}$, $l = -\frac{37}{2}$, $S = -143$; find n and d.
8. Given $a = \frac{11}{2}$, $d = -\frac{3}{4}$, $S = -2$; find n and l.
9. Given a, l, and d; derive a formula for n.
10. Given a, , and n; derive a formula for d.

11. Given a, n, and S; derive a formula for l.

12. Given d, n, and S; derive a formula for a.

13. Given d, l, and n; derive a formula for a and for S.

14. The 6th term of an arithmetic progression is 25 and the 11th term is 5. Find the 30th term.

15. The 5th term of an arithmetic progression is $\frac{7}{15}$, the 21st term is $\frac{11}{3}$, and the last term is $\frac{17}{3}$. Find the number of the terms.

16. The sum of the 2d and the 9th terms of an arithmetic progression is -8; and the sum of the 5th and the 10th terms is $-\frac{8}{3}$. Find the first term.

17. Find four numbers in arithmetic progression such that the sum of the first two shall be -51, and the sum of the last two shall be 9.

18. Find five numbers in arithmetic progression such that the sum of the second, third, and fifth shall be 7; and such that the product of the first and fourth shall be -14.

19. Find three numbers in arithmetic progression such that the sum of the squares of the first two shall be 193; and such that the sum of the first and third shall be 7 less than 3 times the second.

20. Find three integers in arithmetic progression such that their sum shall be 6, and their product -192.

GEOMETRIC PROGRESSION

267. A **geometric progression** (G. P.) is a sequence of numbers called **terms**, each of which, after the first, is derived by multiplying the preceding term by a fixed number called the **ratio**.

Thus, 2, 6, 18, 54, $\cdots$ is a geometric progression. Each term, after the first, is obtained by multiplying the preceding term by 3. The ratio is 3.

Again, $15, -5, +\frac{5}{3}, -\frac{5}{9}, \cdots$ is a G. P. The ratio is $-\frac{1}{3}$. The next two terms are $+\frac{5}{27}$ and $-\frac{5}{81}$.

NOTE. — The ratio may be found by dividing any term by the one preceding it.

EXERCISE 198

Determine which of the following are geometric progressions; determine the ratio and also the next two terms of the geometric progressions:

1. 2, 4, 8, 16, ⋯⋯

2. 64, 32, 16, ⋯⋯

3. 81, 30, 21, ⋯⋯

4. 3, − 6, 12, − 24, ⋯

5. $7n, \frac{7n}{3}, \frac{7n}{9}, \cdots\cdot$

6. $2y, 6y^2, 18y^3, \cdots\cdot$

7. $(m+n), (m+n)^2, (m+n)^3, \cdots\cdot$

8. 3, − 9, − 27, 81

9. $\frac{1}{a}, \frac{1}{a^3}, \frac{1}{a^5}, \cdots\cdot$

10. $\frac{15}{r}, \frac{5}{r^2}, \frac{5}{3r^2}, \cdots\cdot$

Write the first five terms of the G. P. in which

	11	**12**	**13**	**14**	**15**
The first term is	− 3	72	$\frac{1}{4}$	$\frac{1}{3}y$	a
The ratio is	− 2	$\frac{1}{4}$	3	$\frac{1}{2}$	r

268. The *n*th term of a geometric progression can be found without determining all the preceding terms.

Given — the first term, a; the ratio, r; and the number of terms, n, of a geometric progression.

Find — the nth term, l.

Solution. — 1. The progression is $a, ar, ar^2, \cdots\cdot$

2. The exponent of r in each term is 1 less than the number of the term. Hence the 8th term would be ar^7 and the 20th term, ar^{19}.

3. Similarly, the nth term must be ar^{n-1}.

$$\therefore \boxed{l = ar^{n-1}}$$

EXERCISE 199

1. Find the 8th term of 1, 2, 4, ····

2. Find the 6th term of 5, 3, $\frac{9}{5}$, ····

3. Find the 11th term of 4, -12, 36, ····

4. Find the 7th term of $-\frac{3}{2}$, 3, -6, ····

5. Find the 9th term of 5, $\frac{5}{2}$, $\frac{5}{4}$, ····

6. Find the 10th term of $\frac{m}{128}, \frac{m^2}{64}, \frac{m^3}{32}$, ····

7. Indicate the 12th term of 1, $(a+b)$, $(a+b)^2$, ····

8. Indicate the 16th term of 1, $\frac{1}{3}$, $\frac{1}{9}$, ····; also the tth term.

9. Indicate the 12th term of $x, \frac{x^2}{4}, \frac{x^2}{16}, \frac{x^2}{64}$, ····; also the $(n+1)$th term.

10. What term of the progression 5, 10, 20, 40, ···· is 640?

11. What term of the progression 7, 14, 28, ···· is 448?

12. What term of the progression 12, 6, 3, ···· is $\frac{3}{32}$?

13. If the first term of a geometric progression is 4 and the 5th term is $\frac{4}{81}$, what is the ratio?

Find the ratio of the geometric progression if:

14. The first term is 4 and the 5th term is $\frac{1}{64}$.

15. The first term is $\frac{2}{9}$ and the 6th term is 54.

Find by logarithms the value of:

16. The 15th term of the G. P. of which $a = 25$ and $r = 4$.

17. The 20th term of the G. P. of which $a = 1$ and $r = 1.04$.

18. The 30th term of the G. P. of which $a = 50$ and $r = 1.02$.

269. The terms of a geometric progression between any two other terms are called the **geometric means** of those two terms.

Thus, the three geometric means of 2 and 162 are 6, 18, and 54, since 2, 6, 18, 54, 162 form a geometric progression.

A single geometric mean of two numbers is particularly important. It is called **the geometric mean** of the numbers.

When two numbers are given, any specified number of geometric means may be inserted between them.

Example. — Insert three *real* geometric means between 9 and $\frac{16}{9}$.

Solution. — 1. There results a geometric progression of 5 terms, in which $a = 9$, $l = \frac{16}{9}$, and $n = 5$. Find r.

2. $l = ar^{n-1}$. $\therefore \frac{16}{9} = 9 \cdot r^4$, or $r^4 = \frac{16}{81}$.

$$\therefore r = \sqrt[4]{\tfrac{16}{81}} = \pm \tfrac{2}{3}.$$

3. The progression is: $9, 6, 4, \frac{8}{3}, \frac{16}{9}$, or $9, -6, 4, -\frac{8}{3}, \frac{16}{9}$.

NOTE. — The other two fourth roots of $\frac{16}{81}$ are imaginary, and imaginary *means* result from using them.

Check. — Each G. P. has three terms between 9 and $\frac{16}{9}$.

EXERCISE 200

1. Insert 4 geometric means between 1 and 243.
2. Insert 5 geometric means between 1 and 64.
3. Insert 2 geometric means between $\frac{1}{16}$ and 4.
4. Find the geometric mean of 9 and 81.
5. Find the geometric mean of $4y$ and $16y^7$.
6. Find the geometric mean of $4n$ and $\frac{1}{16}n$.
7. Find the geometric mean of $\dfrac{a}{b^2}$ and $\dfrac{b^2}{a}$.
8. Find the geometric mean of x and y.
9. Insert 3 real geometric means between 4 and 16.
10. Insert 2 real geometric means between x and y.
11. Insert 3 real geometric means between a and b.
12. Find by logarithms the ratio if 6 geometric means are inserted between 5 and 50.
13. Find by logarithms the ratio if 4 geometric means are inserted between 15 and 900.
14. Find by logarithms the ratio if 5 geometric means are inserted between 2 and 2000.

270. **The sum of the first n terms of a geometric progression** can be found without writing down the n terms.

Given — the first term, a, the ratio, r, and the number of terms, n, of a geometric progression.

Find — the sum of the n terms, S.

Solution. — 1. $S = a + ar + ar^2 \cdots\cdots + ar^{n-2} + ar^{n-1}$

2. M_r $rS = ar + ar^2 \cdots\cdots + ar^{n-1} + ar^n$

3. $S - rS = a \cdots\cdots - ar^n$

4. $\therefore\ (1 - r)S = a - ar^n.$

5. $D_{(1-r)}$ $\boxed{S = \dfrac{a - ar^n}{1 - r}}$

6. Since $l = ar^{n-1}$, $rl = ar^n$

$\therefore$ $\boxed{S = \dfrac{a - rl}{1 - r}}$

Example. — Find the sum of the first 6 terms of 2, 6, 18, ····

Solution. — 1. $a = 2$, $r = 3$, $n = 6$. Find S.

2. $S = \dfrac{a - ar^n}{1 - 3}$. $\therefore\ S = \dfrac{2 - 2 \cdot 3^6}{1 - 3} = \dfrac{2 - 1458}{-2} = \dfrac{-1456}{-2} = 728.$

EXERCISE 201

Find the sum of the first:

1. Seven terms of the progression 6, 12, 24, ····

2. Eight terms of the progression 20, 10, 5, ····

3. Six terms of the progression 4, -12, $+36$, ····

4. Five terms of the progression $\frac{1}{36}$, $-\frac{1}{12}$, $\frac{1}{4}$, ····

5. Five terms of the progression -4, 20, -100, ····

6. Twelve terms of the progression 1, x^2, x^4, x^6, ····

7. Fourteen terms of the progression 5, $5b^2$, $5b^4$, ····

8. Find the sum of 20 terms of 1, $(1 + r)$, $(1 + r)^2$, ····

9. Find the sum of the first 10 powers of 2.

10. Find the sum of the first 10 powers of 3.

11. *a.* Find the sum of 10 terms of 1, 1.04, $(1.04)^2$, $(1.04)^3$, ⋯.

b. If you have had the chapter on logarithms, find the value of this sum to three decimal places.

12. *a.* Find the sum of 25 terms of 1, 1.025, $(1.025)^2$, $(1.025)^3$, ⋯.

b. By logarithms find the value of this sum.

13. What is the sum of the 7th to the 12th terms, inclusive, of the geometric progression whose first term is 15 and whose ratio is $\sqrt{2}$?

14. If a man were to deposit in a savings bank $5 on the 2d of January, $10 on the 2d of February, $20 on the 2d of March, how much would he have deposited by the end of the year?

15. Suppose a boy were to save 2¢ during the year when he is five years old, 4¢ when he is six years old, 8¢ when he is seven years old, etc. How much would he have when he became 22 years old?

SUPPLEMENTARY TOPIC

271. An **infinite geometric progression** is one the number of whose terms is infinite (uncountable).

If the ratio of the progression is greater than 1 (for example, 1.5), then the terms become larger and larger, and the sum of an infinite number of the terms is infinitely large.

If the ratio of the progression is 1, then every term equals the first term. Again, however, the sum of an infinite number of terms will be infinitely large.

Thus, consider the progression 3, 3, 3, 3, ⋯. In it, the ratio is 1. The sum of a million terms is 3,000,000. The sum of an infinite number of terms is 3 times an infinitely large number; therefore, the sum is infinitely large.

When the ratio is less than 1, however, **the progression is especially interesting.**

Example 1. — Consider the progression $5, \frac{5}{3}, \frac{5}{9}, \cdots$. What happens as the number of terms increases infinitely?

Solution. — 1. The ratio is $\frac{1}{3}$.

2. When $n = 4$, $\quad l = ar^{n-1} = 5(\frac{1}{3})^3 = \frac{5}{27}$.

$$S_n = \frac{a - rl}{1 - r} = \frac{5 - \frac{1}{3} \cdot \frac{5}{27}}{1 - \frac{1}{3}} = \frac{5 - \frac{5}{81}}{1 - \frac{1}{3}}.$$

3. When $n = 10$, $\quad l = 5(\frac{1}{3})^9 = \frac{5}{19{,}683}$.

$$S_n = \frac{a - rl}{1 - r} = \frac{5 - \frac{1}{3} \cdot \frac{5}{19{,}683}}{1 - \frac{1}{3}}$$

$$= \frac{5 - \frac{5}{59{,}049}}{1 - \frac{1}{3}}.$$

4. As n increases, l decreases; also the term rl of S decreases.

5. If n were to become infinitely large, l would become approximately zero; therefore rl also would become approximately zero.

6. $\therefore S_n$ is almost $\frac{5 - 0}{1 - \frac{1}{3}}$ when n is infinitely large.

Consider now the geometric progression in which the first term is a and the ratio, r, is less than 1.

The sum of n terms is $S_n = \frac{a - ar^n}{1 - r}$.

Since r is less than 1, r^n decreases as n increases and becomes approximately zero as n becomes infinitely large.

For example, the 30th power of $\frac{1}{2}$ is $\frac{1}{1{,}073{,}741{,}824}$ and this is certainly almost zero. Moreover, remember this is only the 30th power of $\frac{1}{2}$. How much smaller even the 60th power would be! And an infinitely large power, — how negligibly small it would be!

No matter what a may be, if it is multiplied by a number which is approximately zero, the product also is approximately zero. Therefore ar^n is approximately zero, when n is infinitely large.

$$\therefore\ S_n \text{ becomes approximately } \frac{a - 0}{1 - r} \text{ or } \frac{a}{1 - r}.$$

Hence the sum of an infinite number of terms of a geometric progression in which r is numerically less than 1 is given by the formula

$$S = \frac{a}{1 - r}.$$

Example 2. — What is the sum of an infinite number of terms of $4, -\frac{8}{3}, \frac{16}{9}, \cdots$?

Solution. — 1. $-\frac{8}{3} \div 4 = -\frac{2}{3}$; also $(\frac{16}{9}) \div (-\frac{8}{3}) = -\frac{2}{3}$.

$\therefore r = -\frac{2}{3}$.

2. $\therefore$ the absolute value of r is $\frac{2}{3}$, which is less than 1. $a = 4$

3. $\therefore S = \dfrac{a}{1 - r} = \dfrac{4}{1 + \frac{2}{3}} = \dfrac{4}{\frac{5}{3}} = \dfrac{12}{5}$, or 2.4.

Example 3. — What is the sum of an infinite number of terms of the progression $1, \frac{1}{2}, \frac{1}{4}, \cdots$?

Solution. — 1. $a = 1$; $r = \frac{1}{2}$; $S = \dfrac{a}{1 - r}$.

2. $\therefore S = \dfrac{1}{1 - \frac{1}{2}} = \dfrac{1}{\frac{1}{2}} = 2$.

Check. — Does $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$ approximately equal 2? Yes, as you will easily see if you think of these numbers as 1″, ½″, ¼″, etc.

EXERCISE 202

Find the sum to infinity of:

1. $8, 2, \frac{1}{2}, \cdots$

2. $3, 1, \frac{1}{3}, \cdots$

3. $12, 3, \frac{3}{4}, \cdots$

4. $5, .5, .05, \cdots$

5. $1, \frac{1}{10}, \frac{1}{100}, \cdots$

6. $y, \frac{y}{3}, \frac{y}{9}, \cdots$

7. $m^2, \frac{m^2}{10}, \frac{m^2}{100}, \cdots$

8. $1, -\frac{1}{3}, +\frac{1}{9}, \cdots$

9. $-4, -\frac{1}{2}, -\frac{1}{16}, \cdots$

10. $\frac{1}{5}, -\frac{1}{8}, +\frac{5}{64}, \cdots$

11. Find the sum of the repeating decimal .8181 ⋯

Solution. — 1. $.8181 \cdots = \dfrac{81}{100} + \dfrac{81}{10{,}000} + \text{etc.}$

2. This is a geometric progression in which $a = \dfrac{81}{100}$, and $r = \dfrac{1}{100}$.

The value of the decimal if an infinite number of places is considered is given then by the formula

$$S = \frac{a}{1-r}.$$

3. The value is $\dfrac{\frac{81}{100}}{1-\frac{1}{100}} = \dfrac{81}{100} \div \dfrac{99}{100} = \dfrac{81}{99} = \dfrac{9}{11}.$

Find the value of the following repeating decimals:

12. .6666⋯

13. .4444⋯

14. .636363⋯

15. .8333⋯

16. .409090⋯

17. .3555⋯

18. .1111⋯

19. .24333⋯

20. .52222⋯

XXIII. THE BINOMIAL THEOREM

272. The binomial theorem is a formula for expanding any power of a binomial.

By actual multiplication:

$$(a + x)^2 = a^2 + 2\,ax + x^2. \tag{1}$$

$$(a + x)^3 = a^3 + 3\,a^2x + 3\,ax^2 + x^3. \tag{2}$$

$$(a + x)^4 = a^4 + 4\,a^3x + 6\,a^2x^2 + 4\,ax^3 + x^4. \tag{3}$$

Rule. — To expand any power of a binomial, like $(a + x)^n$:

1. The exponent of a in the first term is n and decreases by 1 in each succeeding term until it becomes 1 in the next to the last term.

2. The first term does not contain x. The exponent of x in the second term is 1 and increases by 1 in each succeeding term until it becomes n in the last term.

3. The coefficient of the first term is 1; of the second is n.

4. If the coefficient of any term be multiplied by the exponent of a in that term, and the product be divided by the number of the term, the quotient is the coefficient of the next term.

Example 1. — Expand $(a + x)^5$.

Solution. — 1. The exponents of a are 5, 4, 3, 2, 1. The exponents of x, starting with 1 in the second term, are 1, 2, 3, 4, and 5. Writing the terms *without the coefficients* gives:

$$a^5 + (?)a^4x + (?)a^3x^2 + (?)a^2x^3 + (?)ax^4 + x^5.$$

2. The coefficient of the first term is 1, and of the second term is 5 (Rule 3). Multiplying 5, the coefficient of the second term, by 4, the exponent of a in the second term, and dividing by 2, the number of the term, gives 10, the coefficient of the third term; and so on.

Filling in the coefficients in this manner gives:

$$(a + x)^5 = a^5 + 5\,a^4x + 10\,a^3x^2 + 10\,a^2x^3 + 5\,ax^4 + x^5.$$

Observe that the number of terms is $n+1$.

Thus, there are 6 terms in Example 1, in which $n = 5$.

Observe also that the coefficients of terms "equidistant" from the ends are the same; for example, the second and the next to the last, etc.

Example 2. — Expand $\left(2 - \frac{m}{3}\right)^6$.

Solution. — 1. In this example, a is 2 and x is $\left(-\frac{m}{3}\right)$.

2. $$\therefore \left(2-\frac{m}{3}\right)^6 = 2^6 + 6\cdot 2^5\cdot\left(-\frac{m}{3}\right) + 15\cdot 2^4\cdot\left(-\frac{m}{3}\right)^2$$
$$+ 20\cdot 2^3\cdot\left(-\frac{m}{3}\right)^3 + 15\cdot 2^2\cdot\left(-\frac{m}{3}\right)^4 + 6\cdot 2\cdot\left(-\frac{m}{3}\right)^5 + \left(-\frac{m}{3}\right)^6$$

3. $$= 64 + 6\cdot 32\left(-\frac{m}{3}\right) + 15\cdot 16\cdot\frac{m^2}{9} + 20\cdot 8\left(-\frac{m^2}{27}\right) + 15\cdot 4\cdot\frac{m^4}{81}$$
$$+ 12\left(-\frac{m^5}{243}\right) + \frac{m^6}{729}$$

4. $$= 64 - 64\,m + \frac{80}{3}\,m^2 - \frac{160}{27}\,m^3 + \frac{20}{27}\,m^4 - \frac{4}{81}\,m^5 + \frac{m^6}{729}.$$

NOTE 1. — When the second term of the binomial is negative, the terms of the expansion are alternately positive and negative.

NOTE 2. — When the terms of the binomial are complicated monomials, place each in parentheses, and afterwards simplify as in Steps 3 and 4.

EXERCISE 203

Expand the following:

1. $(m+n)^5$
2. $(x-y)^4$
3. $(x+1)^5$
4. $(b-2)^4$
5. $(r^2-s^2)^5$
6. $(a-2\,b)^4$
7. $(2\,b+c)^5$
8. $(1+y)^8$
9. $(1-y^2)^6$
10. $(a-4\,b)^4$
11. $(n-\frac{1}{2})^4$
12. $(\frac{1}{3}+a^2)^6$
13. $(2\,x-3)^5$
14. $(ax^2+b)^5$
15. $(4+x^3)^4$

Find the first three terms of:

16. $(m - 4)^{10}$

17. $(x^2 + 2y)^{15}$

18. $(a - \frac{1}{2})^{20}$

19. $(m - n^2)^{10}$

20. $(a + 3b^4)^{16}$

21. $(a^2 - 2b^3)^8$

22. $\left(\frac{x}{a} + \frac{y}{b}\right)^8$

23. $\left(\frac{a}{2} - \frac{b}{3}\right)^6$

24. $\left(\frac{1}{y} - y\right)^6$

25. Find the approximate value of $(1.05)^{10}$ to four decimal places.

Solution. — 1. $(1.05)^{10} = (1 + .05)^{10}$

2. $(1 + .05)^{10} = 1^{10} + 10 \cdot 1^9 \cdot .05 + 45 \cdot 1^8 \cdot (.05)^2 + 120 \cdot 1^7 \cdot (.05)^3$
$+ 210 \cdot 1^6 \cdot (.05)^4$

3. $= 1 + 10 \cdot 1 \cdot .05 + 45 \cdot 1 \cdot .0025 + 120 \cdot 1 \cdot .000125$
$+ 210 \cdot 1 \cdot .00000625^{+} \cdots$

4. $= 1 + .50 + .1125 + .0150 + .0013125$

5. $= 1.6288125$, or 1.6288

Find to four decimal places:

26. $(1.01)^{10}$ **27.** $(1.03)^{15}$ **28.** $(1.04)^{10}$ **29.** $(1.02)^{20}$

30. Write the first four terms of $(a - b)^n$.

SUPPLEMENTARY TOPICS

273. The *r*th or general term of $(a + x)^n$. Following the rules of § 272,

$$(a + x)^n = a^n + n \cdot a^{n-1}x + \frac{n(n-1)}{1 \cdot 2} \cdot a^{n-2}x^2 + \frac{n(n-1)(n-1)}{1 \cdot 2 \cdot 3} a^{n-3}x^3 + \cdots.$$

Note the fourth term. The exponent of x is 1 less than the number of the term; the exponent of a is n minus the exponent of x; the last factor of the denominator equals the exponent of x; in the numerator there are as many factors as there are factors in the denominator. The rule follows.

Rule. — In the rth term of $(a+x)^n$:

1. The exponent of x is $r-1$.

2. The exponent of a is n — the exponent of x, *i.e.*, $n-r+1$.

3. The denominator of the coefficient is $1 \cdot 2 \cdot 3 \cdots (r-1)$, the last factor being the same as the exponent of x.

4. The numerator of the coefficient is $n(n-1) \cdots$ etc., until there are as many factors as in the denominator.

$$\therefore \text{ The } r\text{th term is } \frac{n(n-1)\cdots(n-r+2)}{1 \cdot 2 \cdots (r-1)} \cdot a^{n-r+1} \cdot x^{r-1}.$$

Example. — Find the 8th term of $(3a^{\frac{1}{2}} - b)^{11}$.

Solution. — 1. $(3a^{\frac{1}{2}} - b)^{11} = \{(3a^{\frac{1}{2}}) + (-b)\}^{11}$

2. In the 8th term, the exponent of $(-b)$ will be 7 (Rule 1); the exponent of $(3a^{\frac{1}{2}})$ will be $11-7$, or 4; the last factor of the denominator will be 7, and there will be 7 factors in the numerator starting with $11 \cdot 10$, etc.

3. $\therefore$ The 8th term is $\dfrac{11 \cdot 10 \cdot \overset{3}{\cancel{9}} \cdot \cancel{8} \cdot 7 \cdot \cancel{6} \cdot \cancel{5}}{1 \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5} \cdot \cancel{6} \cdot 7} \cdot (3a^{\frac{1}{2}})^4(-b)^7$

or $$330(81a^2)(-b^7) = -26730a^2b^7.$$

NOTE. — If the second term of the binomial is negative, it should be inclosed, sign and all, in parentheses, before applying the rules. Also, if either term has an exponent or coefficient other than 1, the term should be inclosed in parentheses before applying the rules.

EXERCISE 204

10-4

Find the

1. 5th term of $(m+n)^{10}$
2. 7th term of $(a-b)^9$
3. 10th term of $(m-n)^{15}$
4. 4th term of $(d+3)^8$
5. 9th term of $(b^3+a^2)^{14}$
6. 8th term of $(2x^2-y^4)^{12}$
7. 7th term of $(3x-2)^{10}$
8. 6th term of $(a+\frac{1}{2})^{13}$
9. 9th term of $(a^m-x^n)^{15}$
10. 7th term of $\left(\dfrac{2}{a}-a\right)^{12}$
11. 6th term of $\left(\dfrac{b}{a^2}-\dfrac{a^2}{b^2}\right)^{10}$
12. 7th term of $(d^3-2c^2)^{11}$

274. The binomial formula has not been proved in this chapter; it has been written down from observation of the results in certain special cases. The formula has been applied only for positive integral values of n.

The proof of the formula for positive integral exponents will be found in § 276.

In more advanced courses in mathematics, the formula is proved to be correct (with certain limitations) not only for positive integral values of n but also for negative and fractional values.

Historical Note. — The binomial theorem was formulated by Newton.

275. Compound interest. When interest on a sum of money invested for one interest period is added to the principal, and, with it, draws interest during the next interest period, and so on, then the money is said to be invested at compound interest.

Thus, the interest on \$1 at 4 % for 1 year is \$.04. If this is added to \$1, making \$1.04, and if \$1.04 draws interest for one year, the interest during the second year is \$.0416, and the amount at the end of two years is \$1.04 + .0416 or \$1.0816.

This is called the compound amount, and \$.0816 is called the compound interest. The interest was compounded annually at 4%.

Problem. — Find the compound amount at the end of n years if 1 dollar is invested at rate r compounded annually.

Note. — "Rate r" means 4%, or 5%, or $3\frac{1}{2}$%, etc.

Solution. — 1. The interest during the first year is r, and the amount at the end of year 1 is $1 + r$.

2. The interest for the second year is $r(1 + r)$.

$\therefore$ the amount at the end of year two is $(1 + r) + r(1 + r)$, or $(1 + r)^2$.

3. The interest for the third year is $r(1 + r)^2$.

$\therefore$ the amount at the end of the third year is $(1 + r)^2 + r(1 + r)^2$.

This equals $(1 + r)(1 + r)^2$, or $(1 + r)^3$.

4. Similarly, the compound amount at the end of 4 years is $(1 + r)^4$; at the end of 5 years is $(1 + r)^5$; and at the end of n years is $(1 + r)^n$.

Example. — What is the compound amount of \$250 invested at 4% compounded annually for 10 years?

Solution. — 1. In this example, $r = .04$ and $n = 10$.

2. $\therefore$ the compound amount of \$1 $= (1 + .04)^{10}$.

3. You have found the value of this expression to four decimal places in Example 28, page 399. It is \$1.4802.

4. $\therefore$ the compound amount of \$250 at 4% compounded annually for 10 years is 250×1.4802, or \$370.05.

NOTE. — The value of $(1.04)^{10}$ can be obtained either by the binomial theorem or by logarithms.

EXERCISE 205

Find the compound amount of:

1. \$500 at 4% compounded annually for 10 years.

2. \$500 at 2% compounded annually for 20 years.

3. \$1000 at 6% compounded annually for 5 years.

4. \$1000 at 3% compounded annually for 10 years.

NOTE. — This section gives a mere introduction to the application of the binomial theorem to one of the most important business applications of algebra.

The *mathematics of investment* and *actuarial mathematics* are founded on the mathematics taught in our chapters on exponents, logarithms, progressions (especially geometric), and the binomial theorem. Any student contemplating a course in commerce in any university will find knowledge of these chapters indispensable.

SUPPLEMENTARY TOPICS

276. Proof of the binomial theorem for positive integral exponents. Assume, as in § 272, that

$$(a + x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1 \cdot 2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3^3}a^{n-3}x^3 + \cdots \quad (1)$$

Multiply both members of equation (1) by $a + x$. Then

$$\begin{aligned}(a+x)^{n+1} &= a^{n+1} + na^n x + \frac{n(n-1)}{1 \cdot 2}a^{n-1}x^2 \\ &\qquad + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-2}x^3 + \cdots \\ &\qquad a^n x + \frac{n}{1}a^{n-1}x^2 + \frac{n(n-1)}{1 \cdot 2}a^{n-2}x^3 + \cdots \\ \hline \therefore\ (a+x)^{n+1} &= a^{n+1} + (n+1)a^n x + n\left\{\frac{n-1}{2} + 1\right\}a^{n-1}x^2 \\ &\qquad + \frac{n(n-1)}{1 \cdot 2}\left\{\frac{n-2}{3} + 1\right\}a^{n-2}x^3 \\ &= a^{n+1} + (n+1)a^n x + n \cdot \frac{n+1}{2}a^{n-1}x^2 \\ &\qquad + \frac{n(n-1)}{1 \cdot 2} \cdot \frac{n+1}{3}a^{n-2}x^3 + \cdots \\ &= a^{n+1} + (n+1)a^n x + \frac{(n+1) \cdot n}{1 \cdot 2}a^{n-1}x^2 \\ &\qquad + \frac{(n+1) \cdot n \cdot (n-1)}{1 \cdot 2 \cdot 3}a^{n-2}x^3 + \cdots\end{aligned}$$

Observe that the expansion on the right is in accordance with the rules of § 272. This proves that, if the rules are assumed for any particular positive integer, n, they hold true also for the next greater integer, $n + 1$.

But the rules are known to be satisfactory in the case of $(a + x)^4$; hence they hold for $(a + x)^5$. Since they hold for $(a + x)^5$, then they hold also for $(a + x)^6$; and so on.

Therefore the binomial theorem is true for any positive integer.

277. Fractional and negative exponents. In the expansion of § 272, if n is a positive integer, there is ultimately a term (the $(n + 2)$nd), for and after which the coefficient $\frac{n(n-1)(n-2)\cdots}{1 \cdot 2 \cdot 3 \cdots}$ is zero.

When n is a negative integer or a fraction, there is no term for which the coefficient is zero. Hence the terms continue indefinitely. The resulting expansion has an *infinite number* of terms.

In this case also, the expansion on the right in § 272 has a sum, and this sum is $(a + x)^n$ for any rational (§ 214) value of n, provided the absolute value of a is greater than the absolute value of x. This theorem is proved in a more advanced course in mathematics.

Assuming the theorem, the following examples may be solved:

Example 1. — Expand $(a + x)^{\frac{2}{3}}$ to four terms.

Solution. — 1. Substitute $\frac{2}{3}$ for n in the formula.

2. $\therefore\ (a + x)^{\frac{2}{3}}$

$$= a^{\frac{2}{3}} + \frac{2}{3}a^{\frac{2}{3}-1}x + \frac{\frac{2}{3}(\frac{2}{3} - 1)}{1 \cdot 2}a^{\frac{2}{3}-2}x^2 + \frac{\frac{2}{3}(\frac{2}{3} - 1)(\frac{2}{3} - 2)}{1 \cdot 2 \cdot 3}a^{\frac{2}{3}-3}x^3 + \cdots$$

3. $$= a^{\frac{2}{3}} + \frac{2}{3}a^{-\frac{1}{3}}x + \frac{\frac{2}{3}(-\frac{1}{3})}{2}a^{-\frac{4}{3}}x^2 + \frac{\frac{2}{3}(-\frac{1}{3})(-\frac{4}{3})}{1 \cdot 2 \cdot 3} \cdot a^{-\frac{7}{3}}x^3 + \cdots$$

4. $$= a^{\frac{2}{3}} + \frac{2}{3}a^{-\frac{1}{3}}x - \frac{1}{9}a^{-\frac{4}{3}}x^2 + \frac{4}{81}a^{-\frac{7}{3}}x^3 + \cdots.$$

Example 2. — Find the 7th term of $(a - 3x^{-\frac{3}{2}})^{-\frac{1}{3}}$.

Solution. — 1. The 7th term may be found by applying the rule in § 273.

2. Substitute $(-3x^{-\frac{3}{2}})$ for x, and $(-\frac{1}{3})$ for n.

The exponent of $(-3x^{-\frac{3}{2}})$ is $7 - 1$ or 6.
The exponent of a is $-\frac{1}{3} - 6$ or $-\frac{19}{3}$.
The denominator of the coefficient is $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6$.
The numerator of the coefficient is $(-\frac{1}{3})(-\frac{1}{3} - 1) \cdots$ until there are six factors.

Hence the seventh term is:

$$\frac{(-\frac{1}{3})(-\frac{4}{3})(-\frac{7}{3})(-\frac{10}{3})(-\frac{13}{3})(-\frac{16}{3})}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}(a^{-\frac{19}{3}})(-3x^{-\frac{3}{2}})^6$$

$$= \frac{728}{3^8} \cdot a^{-\frac{19}{3}} \cdot 3^6x^{-9}$$

$$= \frac{728}{9}a^{-\frac{19}{3}}x^{-9}.$$

EXERCISE 206

Find the first four terms of:

1. $(a + x)^{\frac{3}{2}}$
2. $(1 + x)^{-8}$
3. $(1 - x)^{-\frac{3}{4}}$
4. $\sqrt[5]{a - b}$
5. $(a^{\frac{4}{3}} + 2\,b)^{\frac{3}{4}}$
6. $(a^3 - 4\,x^{\frac{1}{2}})^{\frac{1}{2}}$
7. $\dfrac{1}{\sqrt[6]{1 - x}}$
8. $\dfrac{1}{(a + x)^5}$

Find the

9. 6th term of $(a + x)^{\frac{2}{3}}$
10. 5th term of $(a - b)^{-\frac{1}{5}}$
11. 7th term of $(1 + x)^{-7}$
12. 8th term of $(1 - x)^{\frac{3}{5}}$
13. 9th term of $(a - x)^{-3}$
14. 11th term of $\sqrt{(m + n)^5}$
15. 7th term of $(a^{-2} - 2\,b^{\frac{1}{3}})^{-2}$
16. 8th term of $\dfrac{1}{(x^5 + y^{-\frac{1}{2}})^4}$

278. Extraction of roots. The binomial theorem may be used to find the approximate value of a root of a number.

Example. — Find $\sqrt[3]{25}$ approximately to five decimal places.

Solution. — 1. The nearest perfect cube to 25 is 27.

2. $\therefore\ \sqrt[3]{25} = \sqrt[3]{27 - 2} = \{(3^3) + (-2)\}^{\frac{1}{3}}$

3. $= (3^3)^{\frac{1}{3}} + \frac{1}{3}(3^3)^{-\frac{2}{3}} \cdot (-2) - \frac{1}{9}(3^3)^{-\frac{5}{3}}(-2)^2 + \frac{5}{81}(3^3)^{-\frac{8}{3}}(-2)^3 + \cdots$

4. $= 3 - \dfrac{2}{3 \cdot 3^2} - \dfrac{4}{9 \cdot 3^5} - \dfrac{40}{81 \cdot 3^8}$

5. $= 3 - .07407 - .00183 - .00008$, or 2.92402.

Rule. — Separate the given number into two addends, the first of which is the least perfect power corresponding to the order of the root, and expand the result by the binomial theorem.

EXERCISE 207

Find the approximate values to four decimal places of:

1. $\sqrt{17}$
2. $\sqrt{51}$
3. $\sqrt[3]{60}$
4. $\sqrt[3]{75}$
5. $\sqrt[4]{90}$
6. $\sqrt[4]{275}$
7. $\sqrt[5]{35}$
8. $\sqrt[6]{60}$

XXIV. VARIATION

279. In § 139, a **variable** has been described as a *changing* number.

Thus, the distance between a moving train and its destination *varies*, — that is, it decreases; the age of an individual *varies* from moment to moment, — that is, it increases.

280. A quantity which is fixed in any given problem is called a **constant.**

Thus, if a workman receives a fixed sum per day, the total wages due him changes from day to day if he works and remains unpaid. His daily wage is a constant; his total wages is a variable.

281. A change or **variation** in one quantity usually produces a variation in one or more other quantities. Such variables are called **related variables**. For each value of one variable there is a corresponding value of the other variable, or variables.

Thus, if the side of a square is increased, the perimeter and the area of the square are also increased.

Instead of the quantities themselves, their measures in terms of certain units of measure are used.

Thus, distance is expressed as a *number* of miles, rods, or other units of length; weight is expressed as a *number* of units of weight; area is expressed as a *number* of units of area.

282. One quantity **varies directly** as another when the quotient of any value of the one divided by the corresponding value of the other is constant.

Thus, the ratio of the perimeter of a square to the side of the square is always 4, or the perimeter is 4 times the length of the side. If p

represents the perimeter and s represents the length of a side of the square, then $\frac{p}{s} = 4$, or $p = 4s$ expresses the fact that the perimeter varies directly as the side of the square.

Similarly, if x varies directly as y, then $\frac{x}{y} = m$ or $x = my$, where m is a constant.

Thus, since $C = \pi d$ in geometry, we may say, *the circumference of a circle varies directly as the diameter.* This means that the circumference is doubled if the diameter is doubled, or is multiplied by five if the diameter is multiplied by five.

Again, $y = mx^2$ expresses the fact that y varies directly as the square of x. This means, that, if x is *doubled*, y is multiplied by *four;* if x is multiplied by 3, y is multiplied by 9; etc.

283. One quantity is said to **vary inversely** as another when the product of any value of the one and the corresponding value of the other is constant.

Thus, the time and rate of a train going a distance d are connected by the equation $rt = d$. If the distance remains fixed, then the time varies inversely as the rate; for example, if the rate is doubled, the time is halved.

If x varies inversely as y, then $xy = m$, where m is a constant, expresses the relation between them.

284. One quantity is said to **vary jointly** as two others when it varies directly as their product. If x varies jointly as y and z, then $\frac{x}{yz} = m$, where m is a constant, expresses the relation between the variables.

Thus, the wages of a workman varies jointly as the amount he receives per day and the number of days he works; for, letting W equal his total wages, w his daily pay, and n the number of days he works, then $W = nw$. Here $m = 1$.

Again, the formula for the area of a triangle is

$$A = \tfrac{1}{2}ab.$$

This shows that the area of a triangle varies jointly as the base and altitude. (Here $m = \frac{1}{2}$.)

285. Expressing more complicated relations between variables.

x varies directly as y and inversely as z is expressed thus: $\frac{xz}{y} = m$ or $xz = my$, where m is a constant.

x^3 varies inversely as the square of y is expressed thus: $x^3y^2 = m$, where m is a constant.

EXERCISE 208

Express in words the following relations, using the expressions "varies directly," "varies inversely," and "varies jointly."

1. $A = ab$, where $A =$ the area, $a =$ the altitude, and $b =$ the base of a rectangle, and where A, a, and b are regarded as variables.

2. If in Example 1, A and b are variable, and a is constant, express the relation between A and b.

3. If, in Example 1, A is constant, but a and b are variables, express the relation between a and b.

4. $C = \pi r^2$, where C and r are variable and $\pi = 3.1416$.

5. $V = lwh$, where V, l, w, and h are all variable.

6. $V = lwh$, where l is constant, but V, w, and h are variable.

7. $V = lwh$, where l and w are constant, but V and h are variable.

8. $I = PRT$, where I, P, R, and T are all variable.

9. *a.* $I = PRT$, where P and R are constant, but I and T are variable.

b. In this case, if T is trebled, what happens to I?

10. *a.* $I = PRT$, where P is constant, but I, R, and T are variable.

b. In this case, suppose R is doubled and T is trebled, what happens to I?

11. *a.* $V = \frac{4}{3}\pi r^3$, where V = the volume and r = the radius of a sphere, and V and r are considered variable.

b. In this case, if r is doubled, what happens to V?

12. *a.* $nc = C$, where n = a number of articles, c = the cost of each, and C = the total cost, where C is assumed as constant, but n and c are considered variable.

b. In this case, what happens to c if n is doubled?

c. What happens to n if c is doubled?

13. *a.* If, in Example 12, n is constant, but c and C are variable, express the relation.

b. In this case, if C is multiplied by 5, what happens to c?

c. If c is multiplied by 4, what happens to C?

14. *a.* $S = \frac{1}{2}CL$, where S = the lateral area, C = the circumference of the base, and L = the slant height of a right circular cone, where S, C, and L are considered variable.

b. In this example, if C and L are each doubled, what happens to S?

15. The per capita cost of instruction of pupils in a room varies directly as the salary of the teacher, and inversely as the number of the pupils. Select letters to represent the variables, and express this relation by an equation.

16. *a.* The volume of a cone varies jointly as the altitude and the area of the base. Select letters to represent the variables, and express this relation by an equation. (This equation will have in it one letter representing a constant.)

b. Assuming the truth of the equation you have written in part *a*, find the value of the constant if you are told that a cone of base 75, and altitude 5, has volume 125.

17. *a.* The distance a falling body falls from a position of rest varies as the square of the number of seconds it falls. Using suitable letters to represent the variables, express this relation by an equation.

b. If you are told that the distance is 513.28 ft. when the time is 4 seconds, what is the value of the constant you have used in part *a*?

c. Using the constant you have discovered in part *b*, find the distance when the time is 5 seconds.

18. y varies directly as x, and is 40 when x is 4. What is the value of y when x is 9?

19. y varies directly as x^2 and is equal to 60 when x is 3. What is y when x is 5?

20. x varies inversely as y, and is $\frac{2}{3}$ when y is $\frac{3}{4}$. What is the value of x when y is $\frac{3}{2}$?

XXV. DETERMINANTS

286. The symbol $\begin{vmatrix} 3 & 4 \\ 2 & 7 \end{vmatrix}$ is called a **determinant**. Its value is defined to be $3 \cdot 7 - 2 \cdot 4$, which equals $21 - 8$, or 13.

In general $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$ is called a **determinant of the second order** and is defined thus: $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc.$

The numbers a, b, c, and d are called the **elements** of the determinant.

Clearly, any difference such as $rs - mn$ may be arranged as a determinant: thus $rs - mn = \begin{vmatrix} r & n \\ m & s \end{vmatrix}.$

Example 1. — $\begin{vmatrix} 2 & -5 \\ 4 & +3 \end{vmatrix} = 2 \cdot 3 - 4(-5) = 6 + 20 = 26.$

Example 2. — $26 - 15 = 2 \cdot 13 - 3 \cdot 5 = \begin{vmatrix} 2 & 5 \\ 3 & 13 \end{vmatrix}.$

EXERCISE 209

Find the values of:

1. $\begin{vmatrix} 6 & 5 \\ 4 & 2 \end{vmatrix}$

2. $\begin{vmatrix} 5 & 3 \\ 2 & -7 \end{vmatrix}$

3. $\begin{vmatrix} 4 & -2 \\ 6 & 9 \end{vmatrix}$

4. $\begin{vmatrix} 3 & -4 \\ -2 & 7 \end{vmatrix}$

5. $\begin{vmatrix} -5 & 3 \\ 2 & 6 \end{vmatrix}$

6. $\begin{vmatrix} 3a & 4 \\ 2c & 1 \end{vmatrix}$

7. $\begin{vmatrix} 2m & -p \\ 2n & r \end{vmatrix}$

8. $\begin{vmatrix} 3a & 4a \\ 2c & 5e \end{vmatrix}$

Express as determinants:

9. $mn - xy$

10. $2ab - cd$

11. $33 - 14$

12. $6c - 5d$

13. $cd + pq$

14. $3mn + 2rs$

287. Solving simultaneous linear equations by means of determinants.

Consider the system $\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$

Solved by elimination as in § 159,

$$x = \frac{ce - bf}{ae - bd} \text{ and } y = \frac{af - cd}{ae - bd}$$

$$\therefore\ x = \frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}} \text{ and } y = \frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}$$

Notice that the two solutions may be expressed as the quotients of determinants whose terms are the coefficients of the equations.

Rule. — To solve two simultaneous linear equations having two unknowns by determinants:

1. Arrange the equations in the form: $\begin{cases} ax + by = c. \\ dx + ey = f. \end{cases}$

2. The value of x is a fraction: its denominator is the determinant formed by the coefficients of x and y, $\begin{vmatrix} a & b \\ d & e \end{vmatrix}$; its numerator is the determinant obtained by replacing the coefficients of x in the denominator determinant by the corresponding absolute terms, $\begin{vmatrix} c & b \\ f & e \end{vmatrix}$.

3. The value of y is a fraction with the same denominator as x; its numerator is the determinant obtained by replacing the coefficients of y in the denominator determinant by the absolute terms, $\begin{vmatrix} a & c \\ d & f \end{vmatrix}$.

Example. — Solve by determinants the pair of equations:

$$\begin{cases} 2x - 5y = -16. \\ 3x + 7y = 5. \end{cases}$$

Solution. — 1.

$$x = \frac{\begin{vmatrix} -16 & -5 \\ 5 & 7 \end{vmatrix}}{\begin{vmatrix} 2 & -5 \\ 3 & 7 \end{vmatrix}} = \frac{-16 \cdot 7 - 5(-5)}{2 \cdot 7 - 3(-5)} = \frac{-112 + 25}{14 + 15} = \frac{-87}{29} = -3.$$

2.

$$y = \frac{\begin{vmatrix} 2 & -16 \\ 3 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & -5 \\ 3 & 7 \end{vmatrix}} = \frac{10 - 3(-16)}{2 \cdot 7 - 3(-5)} = \frac{10 + 48}{14 + 15} = \frac{58}{29} = 2.$$

Check. — In (1): Does $2(-3) - 5(2) = -16$? Does $-6 - 10 = -16$? Yes.

In (2): Does $3(-3) + 7(2) = 5$? Does $-9 + 14 = 5$? Yes.

EXERCISE 210

1–10. Solve by means of determinants Examples 1–10 of Exercise 109. Check the solutions.

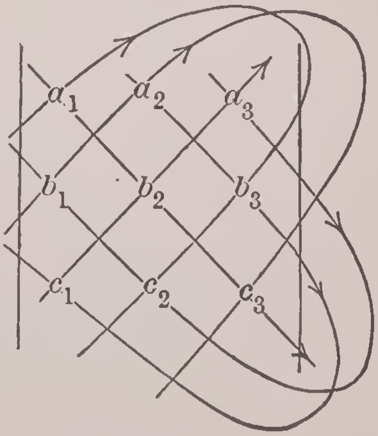
288. Determinants are especially useful in solving simultaneous linear equations with more than two unknowns.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

is called a determinant of the third order. Its value is defined to be:

$$a_1b_2c_3 + a_2b_3c_1 + a_3c_2b_1 - c_1b_2a_3 - b_1a_2c_3 - a_1b_3c_2$$

The adjoining diagram aids in recalling this value. Take the product $a_1b_2c_3$ along the diagonal and add to it the two products formed by starting with a^2 and a^3 respectively and following the arrows which point in the direction of this diagonal; then subtract the product $c_1b_2a_3$ along the other diagonal, and also subtract the two other products formed by starting with b_1 and a_1 respectively and following the arrows which point in the direction of this second diagonal.



Example. — $\begin{vmatrix} 1 & 5 & 2 \\ 4 & 7 & 3 \\ 2 & -3 & 6 \end{vmatrix} = 1 \cdot 7 \cdot 6 + 5 \cdot 3 \cdot 2 + 2(-3) \cdot 4$

$$- 2 \cdot 7 \cdot 2 - 4 \cdot 5 \cdot 6$$
$$- 1 \cdot 3 \cdot (-3)$$
$$= 42 + 30 - 24 - 28 - 120 + 9$$
$$= -91.$$

EXERCISE 211

Find the value of:

1. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 3 & 1 \end{vmatrix}$ **2.** $\begin{vmatrix} 2 & 4 & 6 \\ 3 & -2 & 3 \\ 1 & 5 & 4 \end{vmatrix}$ **3.** $\begin{vmatrix} 2 & 2 & 3 \\ -2 & -4 & -11 \\ 5 & -6 & 2 \end{vmatrix}$

4. Solve by determinants the equations:

$$\begin{cases} 3x + y - z = 14 \\ x + 3y - z = 16 \\ x + y - 3z = -10 \end{cases}$$

Solution. — A rule similar to that of § 287 applies for linear equations with more than two unknowns. Namely:

$$x = \frac{\begin{vmatrix} 14 & 1 & -1 \\ 16 & 3 & -1 \\ -10 & 1 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 1 & 1 & -3 \end{vmatrix}} = \frac{-126 + 10 - 16 - 30 + 48 + 14}{-27 - 1 - 1 + 3 + 3 + 3} = \frac{-100}{-20} = 5.$$

$$y = \frac{\begin{vmatrix} 3 & 14 & -1 \\ 1 & 16 & 1 \\ 1 & -10 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 1 & 1 & -3 \end{vmatrix}} = \frac{-120}{-20} = 6. \qquad z = \frac{\begin{vmatrix} 3 & 1 & 14 \\ 1 & 3 & 16 \\ 1 & 1 & -10 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 1 & 1 & -3 \end{vmatrix}} = \frac{-140}{-20} = 7.$$

Check. — The solution checks when substituted in the three equations.

NOTE. — The equations must be arranged first in the form $ax + by + cz = d$. Thus the equation $2x - 3z = 7$ would be written $2x + 0y - 3z = 7$.

Solve the following equations by determinants:

5.
$$\begin{cases} x + y - z = 24 \\ 4x + 3y - z = 61 \\ 6x - 5y - z = 11 \end{cases}$$

6.
$$\begin{cases} 5x - y + 4z = -5 \\ 3x + 5y + 6z = -20 \\ x + 3y - 8z = -27 \end{cases}$$

7.
$$\begin{cases} 4a - 5b - 6c = 22 \\ a - b + c = -6 \\ 9a + c = 22 \end{cases}$$

8.
$$\begin{cases} 4x - 3y = 1 \\ 4y - 3z = -15 \\ 4z - 3x = 10 \end{cases}$$

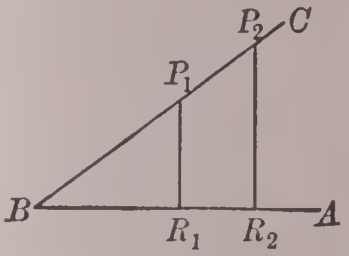
9.
$$\begin{cases} 9x + 5z = -7 \\ 3x + 5y = 1 \\ 9y + 3z = 2 \end{cases}$$

10.
$$\begin{cases} 2x + 5y + 3z = -7 \\ 2y - 4z = 2 - 3x \\ 5x + 9y = 5 + 7z \end{cases}$$

XXVI.

TRIGONOMETRY OF THE RIGHT TRIANGLE

This subject is included as part of the unit in elementary algebra recently proposed by the College Entrance Requirements Board, and by the National Committee on Mathematical Requirements. The following material is designed for those who wish to comply with this requirement. It can be studied at any time after the early work in fractions has been studied, — preferably after the sections devoted to ratio and proportion.



289. Sine of an angle. Carefully make a tracing of the angle ABC of the figure at the right. Extend the sides BA and BC until they are about 5 inches long. (Keep this figure.)

Take point P_1 so that BP_1 is 3 inches. Draw P_1R_1 perpendicular (at right angles) to BA. Measure P_1R_1 to sixteenths of an inch. Find the ratio $\frac{P_1R_1}{BP_1}$ to two decimal places by dividing the length of P_1R_1 by that of BP_1.

Now take P_2 so that BP_2 is 4 inches. Draw P_2R_2 perpendicular to BA. Measure P_2R_2 to sixteenths of an inch. Find the ratio $\frac{P_2R_2}{P_2B}$.

Compare the values you have found for these two ratios. They should be equal at least to one decimal place.

Wherever you place a point P on line BC, when you draw the perpendicular PR, the ratio of it to the distance BP will always have this same value. This ratio is called the **Sine of angle B. (Sin B)**. In the right triangle formed by lines BR, BP, and PR,

$$\sin B = \frac{PR \text{ (\textit{the side opposite angle} } B)}{BP \text{ (\textit{the hypotenuse of the triangle})}}.$$

The value of this ratio changes as the angle changes. The values of the sine for every degree of angle from 1° to 89° inclusive are in the columns headed Sin of the table on page 424. Notice that the sine increases as the angle increases.

Thus $\sin 63° = .891$.

This means that the side opposite a 63° angle of a right triangle, divided by the hypotenuse of the triangle, equals .891.

Again, if $\sin x = .682$, then the angle must be 43°.

The approximate value of the sine of any angle between 1° and 89° can be found by use of this table.

Example. — Find sin 27° 25′.

Solution. — 1.

$$\left.\begin{aligned} \sin 27° &= .454 \\ \sin 27° 25' &= ? \\ \sin 28° &= .469 \end{aligned}\right\} \text{difference} = .015$$

2. $25' = \frac{25}{60}$ or $\frac{5}{12}$ of 1°. $\frac{5}{12} \times .015 = .006$

3. Since the sine increases as the angle increases,

$$\sin 27° 25' = .454 + .006 = .460$$

NOTE. — This process is called **interpolation.**

290. Cosine of an angle. In the figure you drew for § 289 measure BR_1 and also BR_2.

Find: *a.* the ratio $\dfrac{BR_1}{BP_1}$ to two decimals.

b. the ratio $\dfrac{BR_2}{BP_2}$ to two decimals.

Your results, correct to one decimal place, should be the same for the two ratios.

Wherever P is taken on BC, the ratio of BR to BP will have the same value. This ratio is called the **cosine of angle B** (**Cos B**). In the right triangle BPR, BR is the *side adjacent to angle B*, and BP is *the hypotenuse.*

In a right triangle BPR,

$$\text{Cos } B = \frac{BR \text{ (the side adjacent to angle } B)}{BP \text{ (the hypotenuse of the triangle)}}.$$

The value of the cosine of an angle changes as the angle changes. These values are found in the columns headed Cos on page 424 for every degree of angle from 1° to 89°.

Thus, cos 36° = .809.

Notice that **the cosine decreases as the angle increases.**

The approximate value of the cosine of any angle between 1° and 89° can be found by use of this table.

Example. — Find Cos 62° 15′.

Solution. — 1.

$$\left.\begin{array}{lcl} \text{Cos } 62^\circ & = & .469 \\ \text{Cos } 62^\circ\ 15' & = & ? \\ \text{Cos } 63^\circ & = & .454 \end{array}\right\} \text{difference} = .015$$

2. $15' = \frac{1}{4}$ of a degree. $\frac{1}{4}$ of .015 = $.0037^+$, or .004

3. Since the cosine decreases as the angle increases,

$$\cos 62^\circ\ 15' = .469 - .004, \text{ or } .465.$$

291. Tangent of an angle. In the figure you drew for § 289, find:

a. the ratio $\frac{P_1R_1}{BR_1}$ to two decimal places.

b. the ratio $\frac{P_2R_2}{BP_2}$ to two decimal places.

Your results, correct to one decimal place, should agree.

Wherever P is taken, the ratio PR to BR will have the same value. This ratio is called the **tangent of angle B (Tan B).**

In the right triangle BPR,

$$\tan B = \frac{PR \textit{ (side opposite angle } B)}{BR \textit{ (side adjacent angle } B)}.$$

The value of this ratio changes as the angle changes. These values are found in the columns headed Tan on page 424 for every degree of angle from 1° to 89°.

Thus, tan 27° = .510

Notice that the tangent increases as the angle increases.

The approximate value of the tangent of any angle between 1° and 89° can be found by use of this table.

Example. — Find tangent 72° 35′.

Solution. — 1.
$$\left.\begin{array}{ll}\text{Tan } 72^\circ &= 3.078\\ \text{Tan } 72^\circ\ 35' &= \ ?\\ \text{Tan } 73^\circ &= 3.271\end{array}\right\}\text{difference} = .193$$

2.
$$35' = \tfrac{35}{60} \text{ or } \tfrac{7}{12} \text{ of a degree}$$
$$\tfrac{7}{12} \times .193 = \frac{1.351}{12} = .1125, \text{ or } .113$$

3. Since the tangent increases as the angle increases,
$$\text{Tan } 72^\circ\ 35' = 3.078 + .113 = 3.191.$$

EXERCISE 212

1. Obtain from the table:

a. Sin 48°; sin 35°; sin 62°;
b. Cos 34°; cos 67°; cos 46°;
c. Tan 28°; tan 73°; tan 24°.

2. What is the angle x:

a. $\sin x = .438$; $\sin x = .819$; $\sin x = .982$
b. $\cos x = .946$; $\cos x = .766$; $\cos x = .391$
c. $\tan x = .344$; $\tan x = .933$; $\tan x = 2.144$.

3. *a.* Find sin 35° 20′; *d.* sin 20° 15′;
b. Find sin 35° 40′; *e.* sin 76° 25′;
c. Find sin 35° 30′; *f.* sin 43° 50′.

4. *a.* Find cos 40° 10′; *d.* cos 63° 25′;
b. Find cos 40° 20′; *e.* cos 15° 45′;
c. Find cos 40° 30′; *f.* cos 35° 5′.

5. Find *a.* tan 25° 30′ *b.* tan 25° 45′.
6. Find *a.* sin 46° 15′ *b.* sin 46° 35′.
7. Find *a.* cos 48° 20′ *b.* cos 48° 50′.

8. *a.* What is the angle x when $\sin x = .635$?

Solution. — 1.
$$\left.\begin{array}{l}\left.\begin{array}{ll}\sin 39^\circ &= .629\\ \sin x &= .635\end{array}\right\}\begin{array}{l}\text{difference}\\ = .006\end{array}\\ \sin 40^\circ = .643\end{array}\right\}\begin{array}{l}\text{difference}\\ = .014\end{array}$$

2.
$$.006 \div .014 = .42^{+}$$
$$.42 \text{ of } 60' = 25.20' \text{ or } 25'$$

3.
$$\therefore \text{ angle } x = 39^\circ + 25' \text{ (about).}$$

b. What is the angle x when $\sin x = .640$?

Find angle x, when:

9. $\tan x = .350$.

10. $\tan x = 2.165$.

11. $\cos x = .462$.

12. $\cos x = .880$

13. $\sin x = .925$

14. $\tan x = 1.500$

NOTE. — If the value of the sine, cosine, or tangent of x were given, correct to four decimal places, a four place table would be needed to determine the angle x with a corresponding degree of accuracy. However, the angle x can be determined with only slight error in such cases also, by using the three place table.

Thus, if $\sin x = .3148$, find x as if $\sin x = .315$.

292. Finding lengths, distances, and angles by means of the trigonometric ratios.

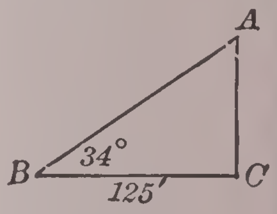
Example 1. — Assume that AC represents a pole of unknown height; that B is 125 ft. from its foot; that the imaginary line AB makes with BC the angle measuring 34°. Find AC.

Solution. — 1. Let $AC = x$ feet. The ratio which uses the sides AC and BC is the tangent of angle B.

$$\text{Tan } B = \frac{AC}{BC}. \quad \therefore \tan 34^\circ = \frac{x}{125}$$

2. $\therefore x = 125 \times \tan 34^\circ$

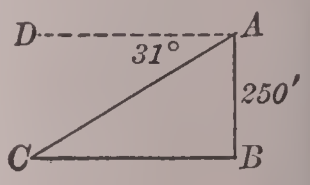
3. $\therefore x = 125 \times .675$, or 84.375 ft., or about 84 ft.



NOTE. — Angle CBA is called the **angle of elevation** of point A at point B.

Example 2. — AB represents a lighthouse 250 ft. high. AD is an imaginary line parallel to BC. C represents the position of a ship. Angle $DAC = 31°$. Find BC.

Solution. — 1. In such a figure, angle BCA equals angle DAC. (Make a tracing of angle DAC on tissue paper, and compare it with angle BCA.) $\therefore$ angle $BCA = 31°$.



2. $\text{Tan } BCA = \frac{AB}{BC}$

3. $\therefore BC \times \tan 31^\circ = 250$

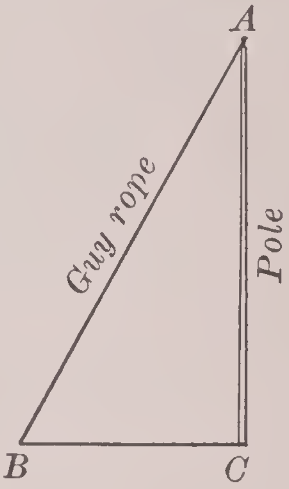
4. $\therefore BC \times .601 = 250$

(tan 31° = .601 in the table.)

5. $D_{.601}$ $BC = 415.9$ ft.

NOTE. — Angle DAC is called the **angle of depression** of point C at point A.

Example 3. — In the adjoining figure, assume the pole to be 40 ft. high and BC 30 ft. How long must the rope AB be made, and what angle does it make with BC?



Solution. — a. To find angle B.

1. $AC = 40$, and $BC = 30$

2. $\text{Tan } B = \frac{AC}{BC}$. $\therefore \text{Tan } B = \frac{40}{30} = 1.333$

3. $$\left.\begin{array}{rl}\left.\begin{array}{r}\text{Tan } 53^\circ = 1.327\\ \text{Tan } B = 1.333\end{array}\right\} & \begin{array}{l}\text{difference}\\ = .006\end{array}\\ \text{Tan } 54^\circ = 1.376 & \end{array}\right\}\begin{array}{l}\text{difference}\\ = .049\end{array}$$

$$\frac{.006}{.049} \times 60' = \frac{360}{49} = 7'_+$$

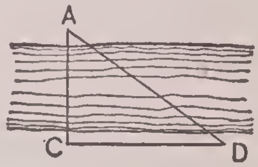
$\therefore$ angle $B = 53^\circ\ 7'$

b. To find AB.

1. $\text{Sin } B = \frac{AC}{AB}$. $\therefore \text{Sin } 53^\circ\ 7' = \frac{40}{AB}$
2. $\therefore AB \cdot \text{Sin } 53^\circ\ 7' = 40$
3. $\therefore .800 \cdot AB = 40$
4. $D_{.800}$ $AB = 50.$

NOTE. — Sin 53° 7′ = .800 is obtained as in Example 3, Exercise 212.

EXERCISE 213



1. In the adjoining figure, AC is perpendicular to CD; $CD = 100$ ft.; angle $D = 62^\circ$. Find AC, and AD.

2. 145 feet from the foot of a high building the angle of elevation of the top is 39°. How high is the building?

3. From the top of a hill known to be 150 ft. above the level of the plain, the angle of depression of a house is 22°. How far away is the house from an imaginary point directly below the top of the hill?

4. A boy is flying a kite at the end of 500 ft. of twine. Directly below the kite a second boy stands. The angle made by the twine (assumed to be stretched straight) with an imaginary line running from the first boy to the second is approximately 50°. Determine the approximate height of the kite.

5. In triangle ABC, BC is 18 ft., angle B is 70°, and AB is 8 ft. Draw a triangle to represent these conditions and draw the altitude AD.

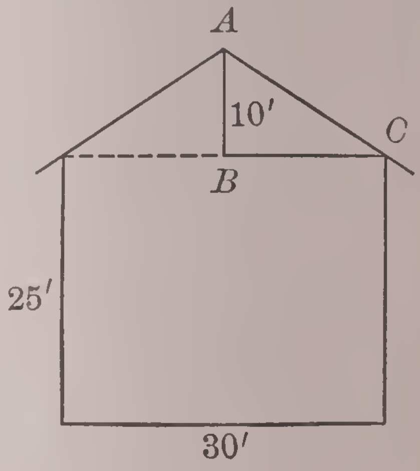
a. Find the length of AD. (Use the sine.)

b. Having found AD, find the area.

6. In the figure at the right is a right triangle in which $AB = 10$ ft. and $BC = 15'$.

a. Find angle C.

b. Find AC.



7. At a point 175 ft. from the foot of a high building, the angle of elevation of the top is 51° 30′. How high is the building?

8. From a height of 350 ft., the angle of depression of an object on the plain below is 28° 15′. Find the distance of the object from a point in the plain directly below the point of observation.

9. In triangle ABC, in which angle C is a right angle, and angle B is an angle of 24° 45′, side AB is 24 in. long. Find the length of AC and of BC.

10. The angle of elevation of an aëroplane at a certain point P is 38° 30′. Point D, 1500 ft. distant, is directly below the aëroplane. How high is the aëroplane?

293. Logarithms simplify the computation.

Example. — Find angle B when $\sin B = \dfrac{265.3}{494.7}$.

Solution. — 1. Find $265.3 \div 494.7$ by logarithms.

$$\begin{aligned} \log 265.3 &= 2.4237 \text{ or } 12.4237 - 10 \\ \log 494.7 &= 2.6943 \qquad \underline{2.6943} \\ \therefore \log (265.3 \div 494.7) &= 9.7294 - 10 \\ \therefore 265.3 \div 494.7 &= .5362, \text{ or } .536 \end{aligned}$$

2. $\therefore \sin B = .536$. Then angle $B = 32° 24'$. (page 419.)

Note. — There are tables which give the logarithms of the sines, cosines, and tangents of angles. Their use simplifies the computation still more. Instruction in their use is more properly a part of a course in trigonometry.

EXERCISE 214

Use logarithms in the solution of the following examples:

Ex. No.	In Triangle *ABC*, in which Angle $C = 90°$				
	BC	*AB*	*AC*	Angle *B*	Angle *A*
1	846	?	625	?	?
2	975	1154	?	?	?
3	?	683	527	?	?
4	?	?	962	36° 30′	?
5	?	378.5	?	42°	?
6	265.8	?	425.3	?	?
7	483.7	1292	?	?	?
8	37.25	68.43	?	?	?
9	?	29.75	18.54	?	?
10	?	361.7	284.3	?	?
11	.75	?	60	?	?
12	85	100	?	?	?
13	?	90	60	?	?
14	?	?	120	62°	?
15	?	150	?	36° 15′	?
16	248	?	?	?	30°
17	?	625	?	48° 30′	?
18	56.5	?	?	29° 45′	?

TABLE OF VALUES OF TRIGONOMETRIC RATIOS

ANGLE	SIN	COS	TAN	ANGLE	SIN	COS	TAN
1°	.018	.999	.018	**45°**	.707	.707	1.000
2°	.035	.999	.035	**46°**	.719	.695	1.036
3°	.052	.998	.052	**47°**	.731	.682	1.072
4°	.070	.997	.070	**48°**	.743	.669	1.111
5°	.087	.996	.087	**49°**	.755	.656	1.150
6°	.105	.994	.105	**50°**	.766	.643	1.192
7°	.122	.992	.123	**51°**	.777	.629	1.235
8°	.139	.990	.141	**52°**	.788	.616	1.280
9°	.156	.988	.158	**53°**	.799	.602	1.327
10°	.174	.985	.176	**54°**	.809	.588	1.376
11°	.191	.982	.194	**55°**	.819	.574	1.428
12°	.208	.978	.213	**56°**	.829	.559	1.483
13°	.225	.974	.231	**57°**	.839	.545	1.540
14°	.242	.970	.249	**58°**	.848	.530	1.600
15°	.259	.966	.268	**59°**	.857	.515	1.664
16°	.276	.961	.287	**60°**	.866	.500	1.732
17°	.292	.956	.306	**61°**	.875	.485	1.804
18°	.309	.951	.325	**62°**	.883	.469	1.881
19°	.326	.946	.344	**63°**	.891	.454	1.963
20°	.342	.940	.364	**64°**	.899	.438	2.050
21°	.358	.934	.384	**65°**	.906	.423	2.144
22°	.375	.927	.404	**66°**	.914	.407	2.246
23°	.391	.921	.424	**67°**	.921	.391	2.356
24°	.407	.914	.445	**68°**	.927	.375	2.475
25°	.423	.906	.466	**69°**	.934	.358	2.605
26°	.438	.899	.488	**70°**	.940	.342	2.747
27°	.454	.891	.510	**71°**	.946	.326	2.904
28°	.469	.883	.532	**72°**	.951	.309	3.078
29°	.485	.875	.554	**73°**	.956	.292	3.271
30°	.500	.866	.577	**74°**	.961	.276	3.487
31°	.515	.857	.601	**75°**	.966	.259	3.732
32°	.530	.848	.625	**76°**	.970	.242	4.011
33°	.545	.839	.649	**77°**	.974	.225	4.331
34°	.559	.829	.675	**78°**	.978	.208	4.705
35°	.574	.819	.700	**79°**	.982	.191	5.145
36°	.588	.809	.727	**80°**	.985	.174	5.671
37°	.602	.799	.754	**81°**	.988	.156	6.314
38°	.616	.788	.781	**82°**	.990	.139	7.115
39°	.629	.777	.810	**83°**	.992	.122	8.144
40°	.643	.766	.839	**84°**	.994	.105	9.514
41°	.656	.755	.869	**85°**	.996	.087	11.430
42°	.669	.743	.900	**86°**	.997	.070	14.300
43°	.682	.731	.933	**87°**	.998	.052	19.081
44°	.695	.719	.966	**88°**	.999	.035	28.636
45°	.707	.707	1.000	**89°**	.999	.018	57.290

SUPPLEMENTARY EXERCISES

TIMED TESTS

SUMMARY

SUPPLEMENTARY EXERCISES

EXERCISE 215

1. By the formula $A = lC$, find l when $A = 422.1$, and $C = 23.45$.

2. By the formula $S = 2\pi rh$, find r when $S = 1188$, $\pi = \frac{22}{7}$, and $h = 21$.

3. By the formula $A = \frac{1}{2}a(b + c)$, find a when $A = 135$, $b = 15$, and $c = 21$.

4. By the formula $S = \frac{1}{2}gt^2$, find g if $S = 257.6$ and $t = 4$. (Remember that $t^2 = t \cdot t$.)

5. By the formula $A = \frac{1}{2}ab$, find a when $A = 403$ and $b = 31$.

6. By the formula $V = lwh$, find h when $V = 3960$, $l = \frac{33}{2}$, and $w = 5$.

7. By the formula $V = \frac{1}{3}\pi r^2 h$, find h when $V = 264$, $\pi = \frac{22}{7}$, and $r = 3$.

EXERCISE 216

1. Find the area of the circle whose radius is 8 in.

2. What is the area of the trapezoid whose altitude is 12 ft., whose upper base is 5 yd., and whose lower base is 7 yd.?

3. What is the simple interest on \$1650 at 7% for 2 yr. 4 mo.?

4. What is the interest on \$2500 at 6% for 3 yr. 9 mo.?

5. $F = W/r$ is a formula used in physics.

a. Find F when $W = 250$ and $r = 5$.

b. Find W when $F = 700$ and $r = 6$.

6. $K = \frac{1}{2}Mv^2$ is a formula used in physics.

Find K when $M = 200$ and $v = 25$.

7. $H = .24\,c^2Rt$ is a formula used in physics.

Find H when $t = 120$, $R = 12$, and $c = 10$.

8. $C = \dfrac{E}{(R + r)}$ is a formula used in the study of electricity.

Find C when $E = 1.08$, $R = 220$, and $r = 10$.

9. $S = P\left(1 + \dfrac{r}{m}\right)^{mn}$ is a formula used in investment mathematics.

Find the value of S when $P = \$2000$, $r = 6\%$, $m = 2$, and $n = 1$.

10. $K = \dfrac{L - l}{l(T - t)}$ is a formula used in the study of heat.

Find the value of K when $L = 232$, $l = 230$, $T = 100$, and $t = 10$.

11. What is the volume of the sphere whose radius is 10 in.?

12. What is the area of the circle whose diameter is 15 in.?

EXERCISE 217

Example 1. Add ax^3 and $2\,bx^3$.

Solution. — 1. These are like terms with respect to x^3 since they have the common factor x^3. The coefficients of x^3 are a and $2\,b$.

2. Therefore, by the rule on page 42,

$$ax^3 + 2\,bx^3 = (a + 2\,b)x^3.$$

Example 2. Add $5(x + y)$ and $-3(x + y)$.

Solution. — These are like terms with respect to $(x + y)$. The coefficients of $(x + y)$ are 5 and -3.

2. $\therefore\ 5(x + y) + [-3(x + y)] = (5 - 3)(x + y) = 2(x + y)$.

Find the sum of:

1. ax, bx, and cx

2. mx, nx, and $-hx$

3. $2\,ax^2$, $-3\,bx^2$, cx^2, $-4\,dx^2$

4. $5\,mx$, $-2\,nx$, $-6\,px$, $+8\,rx$

5. $2(x + y)$, $-5(x + y)$, and $8(x + y)$

6. $-3(m + n)^2$, $+4(m + n)^2$, $-5(m + n)^2$

7. Add $3(x - y) - 4(x - z) + 5(y - z)$, $-2(x - y) + 6(x - z) - 3(y - z)$, and $(x - y) - 2(x - z) + 4(y - z)$

8. Add $6(x+y)^2 - 5(x+y) - 4$, $-4(x+y)^2 + 2(x+y) + 3$, and $+3(x+y)^2 - (x+y) - 2$.

9. Add $8(r+s) + 5(t+s) - 2(r+t)$, $-7(r+s) + 5(t+s) + 3(r+t)$, and $2(r+s) - 9(t+s) - 5(r+t)$.

10. Add $ax + by + cz$ and $dx - ey + fz$.

11. Add $2(x-y) + 3(x+y)$ and $4(x-y) - 5(x+y)$.

12. Add $7\,rs + 6\,mn + 8\,t$ and $-2\,rs + 3\,mn - 10\,t$.

13. Add $13(a-x) - 4(b-x) + 3(c-x)$ and $2(a-x) + 5(b-x) - 7(c-x)$.

14. Add $2\,ax + 3\,by + 4\,cz$ and $dx - 5\,ey + 3\,fz$.

15. Add $ax^2 + bxy + cy^2$ and $Ax^2 - Bxy + Cy^2$.

16. Add $7\,x^{3n} - 5\,x^{2n} + 3\,x^n - 1$, $-2\,x^{3n} + 5 - 3\,x^n + 6\,x^{2n}$, and $-5\,x^{3n} - 4\,x^n + 8\,x^{2n} - 7$.

EXERCISE 218

1. From $ab - bc + ac$ take $ab + bc - ac$.

2. From $4\,x^3 - 10\,x^2 - 18 + 11\,x$ subtract $3\,x^3 - 17\,x + 7\,x^2 - 20$.

3. Subtract $3\,x^3 - 8 + 5\,x - 6\,x^2$ from $5\,x^2 - 11 + 8\,x^2 - 7\,x$.

4. Subtract $2\,x^2 - 3\,xy + y^2$ from $3\,x^2 - 2\,xy + y^2$.

5. Subtract $12\,a^3 - 4\,a - 9$ from $4\,a^3 + 8\,a^2 - 7$.

6. From $x^5 - 4\,x^4y + 3\,x^3y^2 - 2\,x^2y^3 + xy^4 - y^5$ subtract $-2\,x^4y - 2\,x^5 + x^3y^2 - 3\,x^2y^3 - xy^4 + y^5$.

7. From $\frac{5}{6}\,a - \frac{7}{12}\,b - \frac{3}{10}\,c$ take $\frac{1}{3}\,a + \frac{1}{4}\,b - \frac{1}{5}\,c$.

8. Subtract $-\frac{4}{5}\,v + \frac{1}{4}\,w - \frac{5}{6}\,x$ from $\frac{3}{5}\,v - \frac{1}{8}\,w + \frac{5}{3}\,x$.

9. How much more than $2\,x - 3\,y + 4\,z$ is $5\,x - 6\,y + 2\,z$?

10. How much is $2\,m - 3\,n + 11\,k$ greater than $m - 2\,n + 4\,k$?

11. From the sum of $3\,a^2 - 5\,ab - 6\,b^2$ and $2\,a^2 + 7\,ab + 4\,b^2$ subtract $a^2 - 2\,ab + b^2$.

12. From $9x^2 - 3x + 5$ take the sum of $2x^2 + 3x - 4$ and $3x^2 - 7x + 6$.

13. From the sum of $x - y - 3z$ and $2x - 8z$ take the sum of $5y - 4x + 7z$ and $3x - 2y - 5z$.

14. How much more than $3a - 5b$ is 10?

15. Subtract $\frac{5}{8}r - \frac{2}{3}s + \frac{1}{5}t$ from $1\frac{1}{4}r + \frac{1}{2}s - \frac{3}{10}t$.

EXERCISE 219

1. From $3ax$ take $2bx$.

Solution. — $3ax$ and $2bx$ are like with respect to x. After the sign of $2bx$ is changed, the coefficients of x are $3a$ and $-2b$. Using the rule on page 42, the result is $(3a - 2b)x$.

$$\begin{array}{c} 3ax \\ 2bx \\ \hline (3a - 2b)x \end{array}$$

2. Subtract ay from by.

3. Subtract $3mx$ from $4nx$.

4. Subtract $ax + by$ from $cx + dy$.

5. Subtract $ax - by$ from $cx + dy$.

6. Subtract $ax^2 + bxy + cy^2$ from $Ax^2 + Bxy + Cy^2$.

7. From $mx^2 - nx + p$ subtract $rx^2 - nx + p$.

8. From $ax + by - c$ subtract $Ax + By - C$.

9. Subtract $2ax + 3by - 4cz$ from $5Ax + 4By + 6Cz$.

10. Subtract $5(x + y)^2 + 6(x + y) + 9$ from $13(x + y)^2 - 3(x + y) - 5$.

11. Subtract $2(r + s) - 3(t + r) - 6(t + s)$ from $6(r + s) - 4(t + r) - 8(t + s)$.

12. Subtract $2(a + b)^2 - 3(a + b)(a - b) + (a - b)^2$ from $3(a + b)^2 + 4(a + b)(a - b) - (a - b)^2$.

13. From $3x^{2n} - 2x^ny^m + y^{2m}$ subtract $2x^{2n} + x^ny^m - y^{2m}$.

14. From $x^{3n} - x^{2n} + x^n$ subtract $x^{3n} - 2x^{2n} - x^n$.

15. From $2x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$ subtract $x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$.

EXERCISE 220

1. If $A = P + PRT$:

a. Find R when $A = \$5000$, $P = \$4000$, and $T = 3$ yr.

b. Find P when $A = \$3900$, $R = 6\%$, and $T = 5$ yr.

2. If $l = a + (n - 1)d$:

a. Find d when $l = 40$, $a = -5$, and $n = 10$.

b. Find a when $l = 197$, $n = 43$, and $d = 4$.

3. If $S = \frac{na}{2} + \frac{nl}{2}$:

a. Find n if $S = 750$, $a = 16$, and $l = 34$.

b. Find n if $S = 1000$, $a = 28$, and $l = 52$.

4. If $y = ax + b$:

a. Find y when $a = 7$, $x = 8$, and $b = -12$.

b. Find b when $y = 15$, $x = 2$, and $a = 4$.

5. If $S = at - \frac{1}{2}gt^2$:

a. Find a when $S = 3500$, $t = 8$, and $g = 32$.

b. Find a when $S = 3500$, $t = 10$, and $g = 32$.

6. If $T = 2\pi r^2 + 2\pi rh$ $(\pi = 3\frac{1}{7})$:

a. Find T when $r = 14$ and $h = 20$.

b. Find h when $T = 836$ and $r = 7$.

c. Find h when $T = 704$ and $r = 7$.

7. If $T = \pi r^2 + \pi rl$:

a. Find T when $r = 7$ and $l = 10$.

b. Find l when $T = 286$, $\pi = \frac{22}{7}$, and $r = 7$.

EXERCISE 221

Simplify:

1. $6c - \{8c - (10c - 5) - 9\}$

2. $(8t - m) - [9t - (5t - 3m) - 4t]$

3. $(4x - 2y) - (6x - [5x + 7y - 2] - 3y)$

4. $12c - \{6c - [9c + 4] + [5 - 6c]\}$

5. $2n - [3n - \{4n + 4\} - \{-5n - 9\}]$

6. $a - (2a - [3a + 2] - [4a - 5])$
7. $c - [2c - (6a - b) - (c - 5a - 2b)]$
8. $3n - \{4n - (2n + [5n + 6])\}$
9. $5r - \{6r - (7r - [3r - 2])\}$
10. $(8t - 4) - [(5t + 1) + (3t - \{2t - 1\})]$
11. $(11z - 6) - [(3z - 2) - (4z + \{6z - 8\})]$
12. $(2r - 5) - \{-(3r + 2) - (7r - [8r - 1])\}$
13. $m - [n + (3m - 4n)] - [2m - 3n - (m + 3n)]$
14. $(2x - 3y) - (4y - [5x - y] + [6x - 4y])$
15. $3ab - \{(5ab - 8) - (4ab - 16) + (2ab - 7)\}$

EXERCISE 222

Find:

1. $(x^3 + x^2y + xy^2 + y^3)(x - y)$
2. $(m^2 + n^2)(m^2 - n^2)(m^4 + n^4)$
3. $(x - 3y)^2 - (x + 3y)^2$
4. $(a - x)^3 - (a + x)^3$
5. $(3x - 5y)^2 - 5y(x + 5y)$
6. $(x + a + b)(x - a - b)$
7. $(x^2 - 4x + 4)(x^2 + 4x + 4)$
8. $2(a + 2x)(a - 2x) - (a + 2x)^2$
9. $(x^2 + x + 1) \cdot x^n$
10. $(x^n + 3)(x^n + 3)$
11. $(a^n - b^n)(a^n + b^n)$
12. $(x^a + y^b)(x^a + y^b)$
13. $(x^{3n} + 3x^{2n} + 3x^n + 1) \cdot x^n$
14. $(x^{2a} + x^a + 1)(x^a - 1)$
15. $(x^{2a} + x^a + 1)(x^{2a} - x^a + 1)$
16. $(x^a - y^b)(x^a + y^b)(x^{2a} + y^{2b})$

EXERCISE 223

Divide:

1. $2a^4 - 18a^2 - 7a^3 - a + 3$ by $2a + 1$
2. $10a^3 + 33a^2 - 52a + 9$ by $2a^2 + 7a - 9$
3. $x^3 - 5x^2y + 9xy^2 - 9y^3$ by $x - 3y$

4. $m^4 + m^2n^2 + n^4$ by $m^2 + mn + n^2$
5. $m^3 + 2\,m^2 - 9 - 6\,m$ by $m^2 - m - 3$
6. $8\,x^3 - 64\,y^3$ by $2\,x - 4\,y$
7. $a^2 - 2\,ab + b^2 - c^2$ by $a - b + c$
8. $8\,n^4 - 50\,n^2 + 32$ by $4\,n^2 + 6\,n - 8$
9. $6\,x^4 + 13\,x^3 - 70\,x^2 + 71\,x - 20$ by $3\,x^2 - 7\,x + 4$
10. $a^5 - 6\,a^3 + 6\,a^2 - 7\,a + 6$ by $a^3 - 2\,a^2 + a - 2$
11. $r^2 - s^2 - 2\,st - t^2$ by $r - s - t$
12. $8\,a^2 + 60\,bc - 18\,b^2 - 50\,c^2$ by $4\,a + 6\,b - 10\,c$
13. $a^5 - a^4b - ab^4 + b^5$ by $a^2 - 2\,ab + b^2$
14. $8\,m^5 - 14\,m^2 - 18\,m + 21$ by $4\,m^3 + 6\,m - 7$
15. $a^3 - 3\,abc + b^3 + c^3$ by $a + b + c$

EXERCISE 224

Simplify by performing the indicated operations:

1. $\dfrac{E}{\dfrac{r}{a} + r}$

2. $\dfrac{7\,B}{5} \div B\left\{8 - \dfrac{3\,x}{5}\right\}$

3. $a - 4 + \dfrac{2 + 11\,a}{3\,a}$

4. $x^2 + xy + y^2 + \dfrac{2\,y^3}{x - y}$

5. $4\,m - 3 - \dfrac{16\,m^2 - 10}{4\,m + 3}$

6. $m - n - \dfrac{m^3 - n^3}{m^2 + mn + n^2}$

7. $3\,c + 1 - \dfrac{1 - 5\,c}{4\,c - 3}$

8. $\dfrac{2\,m}{m^2 + m - 6} - \dfrac{3\,m}{m^2 + 3\,m - 10}$

9. $\left\{V + 8 \cdot \dfrac{1}{10}\,V\right\}P \div V$

10. $\left\{\dfrac{x^2}{y^2} + 2 + \dfrac{y^2}{x^2}\right\} \div \left\{\dfrac{x}{y} + \dfrac{y}{x}\right\}$

11. $\left(6 + \dfrac{5\,y + 2}{y^2 - 1}\right) \div \left(2 + \dfrac{1}{y - 1}\right)$

12. $\left(m+1-\frac{30}{m}\right)\div\left(m-4-\frac{5}{m}\right)$

13. $\left(a-\frac{2b^2}{a-b}\right)\div\left(a+\frac{b^2}{a+2b}\right)$

14. $\frac{a}{a+b}\div\left(1+\frac{b}{a-b}\right)$

15. $\left(a-\frac{a-x}{1+ax}\right)\div 1+\left(\frac{a^2-ax}{1+ax}\right)$

16. $\left(\frac{2x}{3y}-2+\frac{3y}{2x}\right)\div\left(\frac{2}{y}-\frac{3}{x}\right)$

17. $\left(2x-3+\frac{5x-4}{2x-3}\right)\div\left(\frac{x-2}{7}-\frac{1-x}{4x-6}\right)$

18. $\frac{a-t}{a+r}-\frac{a+r}{a-t}-\frac{t^2-r^2}{(a-t)(a+r)}$

19. $\left(\frac{2}{x^2-3x+2}-\frac{1}{x-1}\right)\div\left(\frac{1}{x^2-3x+2}-\frac{1}{x-2}\right)$

20. $\left(1+\frac{5x+4}{x^2-3x-4}\right)\left(1+\frac{x-16}{x^2-x}\right)\div\left(\frac{x^2+6x+8}{x^2+x}\right)$

21. $\frac{m^2+6mn+9n^2}{m^2+4mn+4n^2}\div\left(\frac{m^2-9n^2}{m^2-4n^2}\div\frac{m^2-mn-6n^2}{m^2+mn-6n^2}\right)$

EXERCISE 225

1. $\frac{7x^2+10x-24}{(x+1)^2}-\frac{5}{x+1}=7$

2. $\frac{6x+7}{6}=x+2-\frac{7x-13}{2(2x+1)}$

3. $\frac{2x-1}{3x-4}=\frac{4x+5}{6x-1}-\frac{4}{3x-4}$

4. $\frac{5a+1}{2a-3}=\frac{a}{2a-3}-3$

5. $\frac{5-x}{x-3}-\frac{1}{3-x}=0$

6. $\frac{3x}{2x-6}-\frac{4}{5x-15}=\frac{1}{2}$

7. $\frac{9}{3a-5}-\frac{1}{a-3}=\frac{2}{a-2}$

8. $\dfrac{3}{x-2} - \dfrac{1}{x+4} = \dfrac{4}{2x-1}$

9. $\dfrac{3a}{2a-5} - \dfrac{3}{2} = \dfrac{7}{3a+1}$

10. $\dfrac{y}{3} + \dfrac{2y^2}{3y-4} = \dfrac{9y-2}{9}$

11. $\dfrac{12}{3-x} - \dfrac{4-x}{1-x} + 1 = 0$

12. $\dfrac{3a+1}{3} - \dfrac{a+2}{2a+5} = a$

13. $\dfrac{5t^2+4t-7}{5t+8} = \dfrac{3t-2}{3} - \dfrac{1}{5}$

14. $\dfrac{x}{7} + \dfrac{5x-4}{3x+1} = \dfrac{2x-5}{14}$

15. $\dfrac{4x-36}{x^2-9} - \dfrac{11}{x-3} = \dfrac{11}{x+3}$

16. $\dfrac{6}{a-4} - \dfrac{a-6}{a+3} = \dfrac{26-a^2}{a^2-a-12}$

17. $\dfrac{y-7}{y^2+3y-28} + \dfrac{1}{y-4} = -\dfrac{2}{y+7}$

18. $\dfrac{17}{2x-16} = \dfrac{9x+17}{x^2-2x-48} - \dfrac{13}{2x+12}$

19. $\dfrac{x+2}{x+3} - \dfrac{x+5}{x+6} = \dfrac{x+3}{x+4} - \dfrac{x+6}{x+7}$

Hint. — Combine the fractions in the first member; also those in the second member. Then clear of fractions.

20. $\dfrac{y-4}{y-5} - \dfrac{y-1}{y-2} = \dfrac{y-5}{y-6} - \dfrac{y-2}{y-3}$

Exercise 226 — Literal Equations

Solve the following for x:

1. $\dfrac{x+5a}{2} - \dfrac{x+a}{4} = 3a$

2. $\dfrac{x-7c}{5} = \dfrac{y+8c}{10} - 2c$

3. $\dfrac{x+11r}{6} - r = \dfrac{10r-x}{3}$

4. $\dfrac{x+11t}{6} + t = \dfrac{10t-x}{3}$

5. $\frac{x - 2a}{5x} = \frac{4a}{x} - 2$

6. $\frac{5x + 4m}{2x} - \frac{11x - 2m}{6x} = 3$

7. $\frac{3x - 5A}{4} - \frac{9x - 7A}{12} = \frac{2}{3x}$

8. $\frac{5x - 4k}{5} - \frac{10x + 9k}{10} = \frac{51}{6x}$

9. $\frac{x}{gp} + a(h - cQ) = ah$

10. $\frac{Nn}{Mm} = \frac{x}{p - x}$

11. $\frac{4a - x}{a - x} - 1 = \frac{12a}{3a - x}$

12. $\frac{5x + b}{2x - 3b} + 3 = \frac{x}{2x - 3b}$

13. $\frac{6x + 5c}{2x^2 - 2cx} + \frac{2c}{x^2 - c^2} = \frac{3x}{x^2 - c^2}$

14. $\frac{2x}{3x - 4t} = \frac{4x + 5t}{6x - t} - \frac{3t}{3x - 4t}$

15. $\frac{9}{3x - 5r} - \frac{2}{x - 2r} = \frac{1}{x - 3r}$

16. $\frac{3}{x - 2s} - \frac{4}{2x - s} = \frac{1}{x + 4s}$

17. $\frac{ax}{x + b} + \frac{bx}{x + a} = a + b$

18. $\frac{a + b}{x} = \frac{b}{x - b} + \frac{a}{x + a}$

19. $\frac{x}{3} + \frac{2x^2}{3x - 4m} = \frac{9x - 2m}{9}$

20. $\frac{5x + t}{5t} - \frac{4x + 7t}{5x + 8t} = \frac{3x - 2t}{3t}$

Exercise 227 — Quadratic Equations

1. $\dfrac{x}{x-4}+\dfrac{1}{5x}=0$

2. $\dfrac{15c+22}{2+c}=\dfrac{16}{2-c}$

3. $\dfrac{3}{s-3}-\dfrac{2}{s-2}=1$

4. $\dfrac{2y+11}{6y-21}=\dfrac{y+17}{5y-15}$

5. $\dfrac{6}{m}=\dfrac{1}{5}+\dfrac{6}{m+1}$

6. $\dfrac{a-3}{a-6}-\dfrac{2}{a-5}=2$

7. $\dfrac{3}{x-1}=\dfrac{4}{3}-\dfrac{x-4}{2x}$

8. $2-\dfrac{a-3}{a-1}=\dfrac{3a}{2a+5}$

9. $\dfrac{2-3t}{4}-\dfrac{2}{t-2}=\dfrac{7}{4}$

10. $\dfrac{6}{4v-3}+8=\dfrac{3-2v}{2v}$

11. $\dfrac{a}{2a-5}=\dfrac{2a+5}{3a+5}$

12. $\dfrac{1}{y-2}-\dfrac{1}{y+2}=\dfrac{4}{3y}$

13. $\dfrac{7}{x+2}-\dfrac{1}{3}=\dfrac{8x-12}{x^2-4}$

14. $\dfrac{2w+3}{5}=2-\dfrac{w-6}{w-4}$

Exercise 228 — Simultaneous Linear Equations

1. $\begin{cases}\dfrac{3m}{8}+\dfrac{n}{4}=\dfrac{17}{8}\\[2ex]\dfrac{m}{3}+\dfrac{n}{12}=\dfrac{4}{3}\end{cases}$

2. $\begin{cases}\dfrac{x}{7}-\dfrac{3y}{14}=-1\\[2ex]\dfrac{x}{2}+\dfrac{7y}{6}=8\end{cases}$

3. $\begin{cases}x+\dfrac{5y-3}{2}=5\\[2ex]x-\dfrac{4y-5}{7}=-2\end{cases}$

4. $\begin{cases}\dfrac{m}{2}-\dfrac{12n-11}{10}=-2\\[2ex]\dfrac{m}{3}+\dfrac{22n-5}{9}=-1\end{cases}$

5. $\begin{cases}\dfrac{g}{2}-\dfrac{2-4k}{7}=-1\\[2ex]\dfrac{11g-5}{12}+\dfrac{k}{2}=-2\end{cases}$

6. $\begin{cases}\dfrac{2x-1}{2}-\dfrac{5y+1}{3}=0\\[2ex]\dfrac{3y-2}{2}-\dfrac{7x-5}{5}=-\dfrac{3}{2}\end{cases}$

7. $\begin{cases}\dfrac{r-s-6}{2}=\dfrac{7}{6}-\dfrac{r+s}{3}\\[2ex]\dfrac{r+s-9}{2}-\dfrac{s-r-6}{3}=0\end{cases}$

8. $$\begin{cases} \dfrac{2a+3b}{5} - \dfrac{2a-3}{2} = 0 \\ \dfrac{8a+5b}{2} - \dfrac{10a-b}{7} = -4 \end{cases}$$

9. $$\begin{cases} \dfrac{3r+5t}{6} - \dfrac{r}{3} = 8 \\ \dfrac{r+t}{3} = 11 - \dfrac{r-t}{4} \end{cases}$$

10. $$\begin{cases} \dfrac{2x-3y}{6} - \dfrac{y}{3} = x \\ \dfrac{x-3}{2} - \dfrac{x-4y}{4} = -\dfrac{17}{4} \end{cases}$$

11. $$\begin{cases} \dfrac{10}{c} - \dfrac{9}{d} = 4 \\ \dfrac{8}{c} - \dfrac{15}{d} = \dfrac{9}{2} \end{cases}$$

12. $$\begin{cases} \dfrac{9}{a} + \dfrac{10}{b} = -1 \\ \dfrac{6}{a} + \dfrac{15}{b} = 1 \end{cases}$$

13. $$\begin{cases} \dfrac{6}{x} + \dfrac{4}{y} = \dfrac{4}{5} \\ \dfrac{9}{x} + \dfrac{5}{y} = \dfrac{7}{10} \end{cases}$$

14. $$\begin{cases} \dfrac{x}{2} + \dfrac{2y}{3} = \dfrac{7a}{6} \\ \dfrac{x}{3} - \dfrac{5y}{6} = b \end{cases}$$

15. $$\begin{cases} \dfrac{x}{b} - \dfrac{y}{a} = 2 \\ \dfrac{x}{a} + \dfrac{y}{b} = \dfrac{3b^2 - a^2}{2ab} \end{cases}$$

16. $$\begin{cases} \dfrac{x+2a}{ab} + \dfrac{y}{b} = 1 \\ \dfrac{x}{a} + \dfrac{y}{b} = -1 \end{cases}$$

17. $$\begin{cases} \dfrac{x-a}{a} + \dfrac{y-a}{b} = 0 \\ x - y = a + b \end{cases}$$

18. $$\begin{cases} \dfrac{x-c}{d} = \dfrac{d-y}{2c} \\ \dfrac{x+y}{c} = 3 \end{cases}$$

19. $$\begin{cases} \dfrac{s-x}{r} = \dfrac{r+y}{s} \\ rx + sy = 0 \end{cases}$$

20. $$\begin{cases} \dfrac{nx-m}{n} = \dfrac{my-n}{m} \\ mx - ny = 0 \end{cases}$$

21. $$\begin{cases} \dfrac{x}{ab} + \dfrac{y}{b} = \dfrac{1}{a} \\ \dfrac{x}{a} + \dfrac{y}{ab} = \dfrac{1}{b} \end{cases}$$

Exercise 229 — Information Problems

First Degree Equations

1. The greatest depth of Lake Superior is about $\frac{7}{6}$ that of Lake Michigan; the greatest depth of Lake Huron exceeds one sixth that of Lake Michigan by 605 feet. The depth of Lake Superior exceeds that of Lake Huron by 265 feet. Find the depth of each.

2. Probably the largest room in the world under one roof is the passenger concourse of the Union Station in Washington, D.C. Its perimeter is 1780 feet. One fifth of its length exceeds its width by 22 feet. Find its dimensions.

3. Ten times the population of the United States in 1820, in millions, exceeded the population in 1910 by 3.8 millions; the population in 1910 exceeded 7 times the population in 1820 by 25 millions. Find the population in both years.

4. Plants feed upon certain plant foods present in the soil, such as potash, nitrogen, and phosphoric acid. A fair crop of potatoes will remove from an acre of ground about 99 pounds of these three foods. The amount of potash removed is 5 times, and the amount of nitrogen is $2\frac{1}{4}$ times that of phosphoric acid. Find the number of pounds of each used.

5. The distance from San Francisco to London *via* New York is 6990 miles. The part of the journey by rail is 50 miles less than $\frac{15}{17}$ of the part by water. Find the part of the journey on land and the part on water.

6. The elevation of Mt. Whitney, in California, the highest point recorded in the United States, is 14,501 feet, measured from sea level. The lowest point of dry land in the United States is in Death Valley, California. If 52 times the elevation of Death Valley be diminished by 45 and the result be increased by the elevation of Mt. Whitney, the sum is zero. Find and interpret the elevation of Death Valley.

7. If the number of states admitted to the Union since its formation by the 13 original states is diminished by 9, the quotient obtained by dividing that number by 2 equals the original number of states. Find the present number of states (1912).

8. I buy some bulbs from a seed store for \$3, paying 75¢ per dozen for one variety, and 50¢ per dozen for another variety. The number of the first variety purchased exceeds the number of the second variety by 18. Find the number of each variety purchased.

9. On the tower of the City Hall of Philadelphia is a statue of William Penn. If the total height of the tower and statue be divided by the height of the statue, the quotient is 14 and the remainder is 29; the height of the statue exceeds $\frac{1}{17}$ of the height of the tower by 7 feet. Find the height of the tower and of the statue.

10. The City Hall of Philadelphia is said to cover a greater area than any other building in the United States. One fifth of its width exceeds one sixth of its length by 13 feet; and one ninth of its length exceeds one tenth of its width by 7 feet. Find its dimensions.

Exercise 230 — Interest Problems

First Degree Equations — One Unknown

1. One sum of money is invested at 5%; a second sum is invested at 6%. If 3 times the first sum exceeds the second sum by \$100, and if the total income is \$155, find the sums invested.

2. A man has \$5000 invested at 4%. How much money must he invest at 6% to make the total income equivalent to 5% on the total amount invested?

3. A man has \$3000 invested at 3.5%, and \$4500 at 4%. How much must he invest at 6% to make the total income equivalent to 5% on the total sum invested?

4. A man has \$3000 invested part at 5% and part at 6%. His total income per year is \$157. How much has he invested at each rate?

5. The income at 5% on one sum of money exceeds by $35.50 the income at 4% on a sum which is $350 less than the first. Find the two sums invested.

6. The income on one sum of money at $4\frac{1}{2}\%$ and the income on a sum $600 greater at $3\frac{1}{2}\%$ together amount to $421 per year. Find the total amount invested.

Hint. — $4\frac{1}{2}\% = \frac{9}{2}\% = \frac{9}{200}$.

7. A man invests a sum of money in $4\frac{1}{2}\%$ stock and a sum $180 greater than the first in $3\frac{1}{2}\%$ stock. If the incomes from the two investments are equal, find the sums invested.

8. A man has $8000 invested at 5% and $5000 at 7%. How much must he invest at 8% to make his income equal to 6% on his total investment?

First Degree Equations — Two Unknowns

9. A man has $2500 invested from which he receives a total income of $135. Part of the money is invested at 6% and part at $4\frac{1}{2}\%$. How much is invested in each way?

10. A man has all together $5000 of savings. He has part invested in a 5% bond, and the balance invested in a mortgage drawing 6%. If his total income is $280, how much has he invested in each way?

11. A man has $1200 invested at one rate of interest and $500 at a rate which is 1 per cent greater than the former rate. The income from the first investment exceeds the income from the second investment by $23. Find the rate at which each sum is invested.

12. The simple interest on $800 at 5% for a certain number of years exceeds the simple interest on $300 at 6% for a second period of years by $60. If the second period of years exceeds the first by 4 years, find the number of years each sum is invested.

Exercise 231 — Age Problems

First Degree Equations — One Unknown or Two Unknowns

1. A father is now 9 times as old as his son. In 9 years he will be only 3 times as old as his son. What are their present ages?

2. The difference between the present ages of a father and son is 25 years. In 10 years the father will be twice as old as his son. What are their present ages?

3. A is 5 times as old as B. In 9 years he will be only 3 times as old as B. What are their ages?

4. B is twice as old as A. 35 years ago he was 7 times as old as A. What are their present ages?

5. A is 68 years of age, and B is 11. In how many years will A be 4 times as old as B?

6. The age of a father is 5 times that of his son; his age 5 years from now will exceed 3 times his son's age by 4 years. Find their present ages.

7. A's age is three eighths of B's, and 8 years ago it was two sevenths of B's age. Find their ages at present.

8. Washington was admitted to the Union 18 years before Oklahoma, and may therefore be said to be 18 years older than Oklahoma as a state. One fourth of Washington's age in 1911 exceeded Oklahoma's age by $1\frac{1}{2}$ years. Find the year when each was admitted.

9. In 1912, the "age" of Maine was $4\frac{2}{11}$ times that of Wyoming; in 1920, it will be $3\frac{1}{3}$ times it. Find when each state was admitted to the Union.

10. A's age 11 years from now divided by his age 11 years ago is the fraction $\frac{16}{5}$. Find his present age.

11. A's age exceeds twice B's age by 7 years. The sum of three fifths of B's age 2 years ago and of four sevenths of A's age 4 years from now is 26 years. Find their present ages.

12. A's age is three fifths of B's age; but in 16 years A's age will be five sevenths of B's age. Find their ages at present.

13. Three years ago A's age was $\frac{4}{3}$ of B's age; but in nine years his age will be $\frac{11}{9}$ of B's age. Find their present ages.

14. If the age of a university is reckoned from the date of its founding, then, in 1912, the age of Yale exceeded the age of Princeton by 46 years. Four years later, one half of the age of Princeton will exceed one third of the age of Yale by 13 years. Find when each was founded.

15. In 1912 the sum of the ages of the oldest English university, Oxford, and the oldest American university, Harvard, was 1316 years; the age of Oxford exceeded three and three fourths times the age of Harvard by 5 years. Find when each was founded.

Exercise 232 — Geometry Problems

1. In a triangle commonly used by draughtsmen, the second angle is two thirds of the first, and the third angle is one half of the second. Find the angles of the triangle. (§ 13)

2. In another triangle used by draughtsmen, there are two equal angles, each of which is one half of the third angle. Find the angles of this triangle.

3. Probably the largest room in the world under one roof is the passenger concourse of the Union Station in Washington, D.C. Its perimeter is 1780 feet. One fifth of its length exceeds its width by 22 feet. Find its dimensions.

4. The length of the foundation of the Capitol in Washington exceeds twice the width by $51\frac{1}{3}$ feet. The perimeter of the foundation is $2202\frac{2}{3}$ feet. Find the dimensions of the foundations of the Capitol.

5. Twice the width of the Pennsylvania Station in New York exceeds its length by 80 feet. Four times the length exceeds the perimeter by 700 feet. Find the dimensions.

6. Find the three sides of a triangle if the perimeter is 45 inches, if the second side is twice the third side, and if the first side exceeds the third by 5 inches.

7. The base of a certain rectangle exceeds its altitude by 8 inches. If the base and altitude are both decreased by 2 inches, the old area exceeds the new by 36 square inches. Find the dimensions of the rectangle.

8. The base of a rectangle is 9 feet more and the altitude is 8 feet less than the side of a square. The area of the rectangle exceeds the area of the square by 15 square feet. Find the dimensions of the rectangle.

9. A man planned a house whose length exceeded its width by 10 feet. He found that it would be too expensive to build the house as planned, so he decided to decrease both dimensions 5 feet. This decreased the area of the house 425 square feet. What were the original and the new dimensions?

10. A man planned to set out an apple orchard with two more trees to each row than he had rows, but found that that plan left one tree over. He found that if he decreased the number of rows by 3, and increased the number of trees per row by 5, he used all of his trees. How many trees had he?

Quadratic Equations — One Unknown

11. The length of a certain rectangular farm is three times its width. If its length should be increased by 20 rods, and its width by 8 rods, its area would be trebled. Of how many square rods does the farm consist?

12. The standard size city lot in parts of Chicago is five times as long as it is wide. The lots in parts of Indianapolis are 10 feet wider and 5 feet longer than those in Chicago. Three times the area of a Chicago lot exceeds twice the area of an Indianapolis lot by 275 square feet. Find the dimensions of the lots in both cities.

13. An architect who has made plans for a house with a base 30 × 42 feet finds that he must reduce the size. By what equal amount must he reduce the two dimensions of the house in order to make the area of the new base 925 square feet?

14. Find the angle such that 3 times its complement increased by two thirds of its supplement equals 137°.

15. The supplement of a certain angle divided by its complement gives as quotient $2\frac{1}{2}$. What is the angle?

16. The width of a room is three fifths of its length; if 12 feet be added to the width and taken from the length, the room will be a square. Find its dimensions.

17. Five lines radiate from a point making angles such that the second is one half of the first, the third is twice the first, the fourth is the sum of the second and third, and the fifth is three times the third. Find the angles.

18. Find the three angles of a triangle if the second angle is one half of the remainder obtained by diminishing the first angle by 1°, and if the third angle is $\frac{4}{3}$ of the remainder obtained by diminishing the first angle by 7°.

First Degree Equations — Two Unknowns

19. The City Hall of Philadelphia is said to cover a greater area than any other building in the United States. One fifth of its width exceeds one sixth of its length by 13 feet; and one ninth of its length exceeds one tenth of its width by 7 feet. Find its dimensions.

20. The perimeter of a certain isosceles triangle (see p. 102) is 140 inches. Each side exceeds the base by 10 inches. Find the three sides of the triangle.

21. Twice the shorter side of a parallelogram exceeds the longer side by 5 inches; one third of the sum of the shorter side and 9 exceeds one fifth of the longer side by 3. Find the sides of the parallelogram.

22. If a rectangular lot were 6 feet longer and 5 feet wider than it is now, it would contain 839 square feet more; if it were 4 feet longer, and 7 feet wider, it would contain 879 square feet more. Find its length and width.

23. If a field is made 5 feet longer and 7 feet wider, its area would be increased by 830 square feet; but if its length is made 8 feet less, and its width 4 feet less, its area is diminished by 700 square feet. Find its length and width.

24. There are two supplementary angles such that $\frac{1}{3}$ of the larger exceeds $\frac{2}{5}$ of the smaller by 5°. Find the angles.

25. One angle of a triangle is 35°. If the number of degrees in one of the remaining two angles is divided by the number in the other, the quotient is 9 and the remainder is 10. Find the three angles of the triangle.

Quadratic Equations — Two Unknowns

26. Find the dimensions of a rectangle whose area is 357 square feet if its length exceeds its width by 4 feet.

27. The main waiting room of the Union Railway Station in Washington, D.C., has an area of 28,600 square feet. The length exceeds the width by 90 feet. Find the dimensions.

28. The area of a rectangular field is 216 square rods, and its perimeter is 60 rods. Find the length and width of the field.

29. The altitude of a certain rectangle is 2 feet more than the side of a certain square; the perimeter of the rectangle is 7 times the side of the square, and the area of the rectangle exceeds twice the area of the square by 32 feet. Find the side of the square and the base of the rectangle.

30. Find the sides of a parallelogram if the perimeter is 24 inches and the sum of the squares of the number of inches in the long and short sides is 80.

31. One of two angles exceeds the other by 5°. If each is multiplied by its supplement, the product obtained from the larger of the given angles exceeds the other product by the square of the smaller of the given angles. Find the angles.

32. Two angles are supplementary. The square of the number of degrees in the larger angle exceeds by 4400 the product of the number of degrees in one angle by the number in the other angle. Find the number of degrees in each angle.

33. A man has two square lots of unequal size, together containing 74 square rods. If the lots were side by side, it would require 38 rods of fence to surround them in a single inclosure of six sides. Find the length of the side of each.

Exercise 233 — Distance, Rate, and Time Problems

1. Suppose that two men, who travel at the rate of 6 miles and 10 miles per hour respectively, start from the same place in opposite directions. In how many hours will they be 200 miles apart?

2. Suppose that A, traveling 10 miles per hour, leaves a place 3 hours before B; suppose that B travels 15 miles per hour. In how many hours will B overtake A?

3. Suppose A, traveling 15 miles per hour, starts 4 hours before B. At what rate must B travel to overtake A in 10 hours?

4. Two hours after A left, B starts after him in an automobile at the rate of 27 miles an hour and overtakes him in $2\frac{1}{2}$ hours. At what rate was A traveling?

5. A and B travel toward each other from points separated by 250 miles, A at a rate which exceeds B's rate by 8 miles an hour. If they meet in 5 hours, at what rate did each travel?

6. Some boys who are boating on a river know that they can go with the current 6 miles per hour and can return against the current at the rate of 3 miles per hour. How far may they go if they have only 3 hours for the trip?

7. A man has 11 hours at his disposal. How far may he go in a buggy at the rate of 10 miles an hour if he plans to return at an average rate of 7 miles per hour?

8. An automobile is traveling at the rate of 25 miles an hour. In how many hours will a second automobile overtake the first if the second starts 2 hours later than the first, and travels at the rate of 35 miles an hour?

9. An express train whose rate is 36 miles an hour starts 54 minutes after a slow train and overtakes it in 1 hour and 48 minutes. What is the rate of the slow train?

10. An automobile party is traveling at the rate of 20 miles per hour. At what rate must a second automobile travel in order to overtake the first if it starts 2 hours after the first and wishes to overtake it in 3 hours?

11. Chicago and Madison, Wisconsin, are about 140 miles apart. Suppose that a train starts from each city toward the other, one at the rate of 35 miles per hour and the other at the rate of 40 miles per hour. How soon will they meet?

12. The rate of an express train is three times that of a slow train. It covers 180 miles in 8 hours less time than the slow train. Find the rate of each train.

13. A messenger starts out to deliver a package to a point 24 miles distant, at the rate of 8 miles per hour. At what rate must a second messenger travel to arrive at the same time as the first messenger, if he starts 1 hour after him?

14. The rate of a passenger train exceeds twice the rate of a freight train by 5 miles per hour. It can go 350 miles while the freight train goes 150 miles. Find the rate of each train.

15. Two men, A and B, 57 miles apart, travel towards each other, B starting 20 minutes after A. A travels at the rate of 6 miles an hour and B at the rate of 5 miles an hour. How far will each have traveled when they meet?

16. The rate of an express train is five thirds of that of a slow train. It travels 36 miles in 32 minutes less time than the slow train. Find the rate of each train.

17. The rate of an express train is three halves that of a slow train. It covers 270 miles in 3 hours less time than the slow train. Find the rate of the trains.

Linear Equations — Two Unknowns

18. An express train travels 30 miles in 27 minutes less time than a slow train. If the rate of the express train were $\frac{5}{4}$ as great, and if the rate of the slow train were $\frac{4}{5}$ as great, the express train would travel 30 miles in 54 minutes less time than the slow train. Find the rate of each train in miles an hour.

19. The fastest train on the Pennsylvania R. R. makes the trip between Chicago and Fort Wayne, Indiana, a distance of 148 miles, in 1 hour and 20 minutes less time than one of the ordinary trains. Its rate is $\frac{3}{2}$ that of the ordinary train. Find the rate of each train.

Quadratic Equations

20. A man traveled 105 miles. If he had gone 9 miles more an hour, he would have performed the journey in $1\frac{1}{2}$ hours less time. Find his rate in miles an hour.

21. If a man travels 120 miles by one train and returns on a train whose rate is 10 miles an hour more, he will require 7 hours for the trip. What is the rate of the first train?

22. An automobile made a trip of 50 miles, 10 miles within city limits and 40 miles outside city limits. Outside city limits, the rate was increased 15 miles an hour. If the trip took $2\frac{3}{5}$ hours, at what rate did the car travel within and outside city limits?

Exercise 234 — River Problems

Linear Equations — Two Unknowns

1. Some boys who can row 4 miles an hour in still water made a trip on a river whose current is 2 miles an hour. If it took them 8 hours for the trip, how far did they go?

2. A party took a trip in a motor boat which runs at the rate of 15 miles an hour. They took three hours for the trip. What distance did they go, if the rate of the current is 3 miles an hour?

3. Some boys row on a river whose current is known to be $2\frac{1}{2}$ miles per hour. They find that it takes them as long to go upstream 2 miles as downstream 7 miles. What is their rate of rowing?

4. Some boys are rowing on a river whose current is 5 miles per hour in one stretch and 3 miles in another. They find when going downstream that they can go 4 miles where the current is rapid in the same time that they can go 3 miles where the current is slower. Find the rate at which they row in still water.

5. A crew can row 10 miles downstream in 50 minutes, and 12 miles upstream in an hour and a half. Find the rate in miles an hour of the current, and of the crew in still water.

6. A motor boat which can run at the rate of 15 miles an hour in still water went downstream a certain distance in 4 hours. It took 6 hours for the trip back. What was the distance and the rate of the current?

7. A crew is rowing on a stream the rate of whose current is known to be 2 miles an hour; they find that it takes them one and one third hours to go down, and four hours to come back a certain distance. Find the distance and the rate of the crew in still water.

8. A crew can row 8 miles downstream and back again in $4\frac{4}{5}$ hours; if the rate of the stream is 4 miles an hour, find the rate of the crew in still water.

9. Some boys are canoeing on a river in part of which the current is 5 miles an hour, and in another part 3 miles an hour. If, when going downstream, they go 4 miles where the current is rapid and 8 miles where it is less rapid in a total time of $1\frac{5}{6}$ hours, what is their rate of rowing in still water?

Exercise 235 — Digit Problems

1. The tens' digit of a number exceeds its units' digit by 4. If the digits be reversed, the new number is 6 more than one half of the old number. Find the number.

2. The sum of the two digits of a number is 9. If the digits be reversed, the quotient of the new number divided by the units' digit of the given number is 13 and the remainder is 1. Find the number.

3. The sum of the two digits of a number is 16; and if 18 be subtracted from the number, the remainder equals the number obtained by reversing the digits. Find the number.

4. If the digits of a number of two figures be reversed, the sum of the resulting number and twice the given number is 204; and if the given number is divided by the sum of its digits, the quotient is 7 and the remainder is 6. Find the number.

5. If a certain number be divided by the sum of its two digits, the quotient is 4 and the remainder is 3. If the digits be reversed, the sum of the resulting number and 23 is twice the given number. Find the given number.

6. The sum of the two digits of a number is 15. If the digits be reversed, the new number exceeds two thirds of the original number by 35. What is the number?

TIMED TESTS

Purposes. — To afford a ready means:

a. Of comparing individual pupils with the class;

b. Of comparing one class with another;

c. Of stimulating pupils to work more rapidly, and to work more accurately when hurried.

Note. — If the following sets of examples are not used as "timed tests," they will be found useful as so many extra drill examples.

How to be used. — 1. Urge pupils not to write any answers in the text.

2. Tell them to start promptly and to stop at once when the word is given. Explain the purpose of limiting the time.

3. Tell them that both speed and accuracy are important. Tell them that it is not expected that many, or even any in some cases, will complete all the examples in the time allowed.

4. Have the answers ready in advance so that you can give your whole attention to watching the pupils and the clock. Stop the class when the allotted time has expired.

5. Have the papers marked at once by the pupils. If necessary, have the pupils exchange papers. Read the answers once, slowly and distinctly.

6. Have indicated on each paper: *a. The number of examples tried. b. The number of examples right. c. The percentage which the latter is of the former.* (*I.e.*, if a pupil tried 15 examples out of 20 examples, and got 10 right, his accuracy mark is $\frac{10}{15}$ of 100% (66.6%) and not $\frac{10}{20}$ of 100% (50%).

7. Derive the **median speed** of the class as follows:

Direct all who tried only 1 to stand and remain standing; similarly those who tried only 2, only 3, etc., **until one half the class is standing. The number of examples tried by the last pupil to rise is the median speed of the class for the allotted time.**

8. Derive the **median accuracy percentage** of the class as follows:

Direct all who made 100% (see 6 above) to stand and remain standing; similarly those who made 99%, 98%, etc., **until one half the class is standing. The mark of the last pupil to rise is the median accuracy mark of the class.**

9. Record these medians on a sheet to be made out in duplicate (if you will) as follows:

RECORD OF WELLS AND HART TIMED TESTS

School .... Teacher ....

Number of pupils........ Teacher's next summer address..........

TEST NO.	DATE GIVEN	MEDIAN SPEED	MEDIAN ACCUR'CY	NUMBER OF 100% PUPILS	DATE GIVEN	MEDIAN SPEED	MEDIAN ACCUR'CY	NUMBER OF 100% PUPILS
I								
II								
Etc.	Provide space for twenty tests.							

Keep one copy yourself. At the close of the school year, send the other copy to Professor W. W. Hart, University of Wisconsin. Early in the summer he will send you a statement showing the averages of the medians for each test, as reported to him.

10. **Most important of all.** — Assign special practice to all pupils who fall below the medians in speed or accuracy. Provide opportunity for them to demonstrate that they have attained the medians of the class.

Table I (Used in Tests I, II, and III)

1.	**2.**	**3.**	**4.**	**5.**	**6.**	**7.**	**8.**	**9.**	**10.**
+ 3	− 6	− 2	+ 6	+ 8	+ 3	+ 2	+ 8	− 6	− 3
− 9	+ 4	− 9	− 3	− 2	− 7	+ 7	− 5	− 5	− 8

11.	**12.**	**13.**	**14.**	**15.**	**16.**	**17.**	**18.**	**19.**	**20.**
− 7	− 8	− 6	+ 4	− 3	− 7	+ 9	− 8	− 6	− 8
− 5	+ 2	− 3	− 9	+ 9	− 2	− 6	+ 3	− 4	− 7

21.	**22.**	**23.**	**24.**	**25.**	**26.**	**27.**	**28.**	**29.**	**30.**
− 5	+ 9	− 3	− 7	+ 9	+ 7	− 9	− 7	− 9	+ 8
− 8	− 4	− 7	+ 6	− 2	− 9	− 8	+ 8	− 5	− 4

31.	**32.**	**33.**	**34.**	**35.**	**36.**	**37.**	**38.**	**39.**	**40.**
+ 6	− 7	+ 9	− 9	− 8	+ 5	+ 8	− 7	− 9	+ 9
− 5	− 9	− 8	− 6	− 4	− 7	− 8	− 6	− 9	− 5

Note. — For Tests I–III, have the pupils prepare in advance a sheet containing in one column the numbers 1 to 20, written about one half inch apart; and, in a second column, the numbers 21 to 40. Tell them they are to write beside each number the answer to an example having the same number.

Test I — *Addition of Signed Numbers*

Time: 3 Minutes

Add the numbers in Examples 1 to 40. Write only the answers.

Test II — *Subtraction of Signed Numbers*

Time: 3 Minutes

Subtract the lower number from the upper in Examples 1 to 40. Write only the results.

Test III — *Multiplication of Signed Numbers*

Time: 3 Minutes

Multiply the two numbers in Examples 1 to 40. Write only the products.

Test IV — *Evaluation of Expressions*

Time: 10 Minutes

Find the numerical value of the following expressions when

$$a = 4,\ b = 5,\ c = 3,\ d = 4,\ m = 5,\ x = 2.$$

1. $3a + 2b$
2. $5c - 3d$
3. $m^2 + mx$
4. $x^3 - x^2 + 1$
5. $8a - 3b$
6. $9c + 2d$
7. $2m^2 + 3mx^2$
8. $2x^3 + 3x^2 - 5$
9. $\frac{1}{2}ab$
10. $\frac{1}{2}a(b + c)$
11. $2 \cdot \frac{22}{7}mx$
12. $4 \cdot \frac{22}{7} \cdot x^2$
13. $\frac{3}{4}ab^2$
14. $2cd + 3bc$
15. $m^3 - m^2x$
16. $x^3 - 2x^2 + x$
17. $a^2 + 2ab$
18. $5c^2 - 3cd$
19. $2mx + 3mx^2$
20. $x^4 - 2x^3 + x^2$
21. $ax^2 - bx + c$

Table II (Used in Tests V, VI)

1. $7a - 3b + 4c$, and $5a - 7b + 2c$
2. $6r - 7s + t$, $8r - 3s$, and $2r + 5t$
3. $5x - 7y - 9z$, $3y + 4z$, and $-2x + 5z$
4. $-3c + 4d$, $9c - 10d + 3e$, and $-4c + 3e$
5. $12p - 7r$, $-8p + 5r - 3s$, and $7p - 3r - s$
6. $x^2 - 3x + 6$, $-2x^2 + 7$, and $3x^2 + 5x - 8$
7. $8y^2 - 2y + 4$, $-6y - 7$, and $-5y^2 + 6$
8. $3m^2 - 6 + 7m$, $-4m + 10$, and $5m - 7$
9. $-4t^2 + 6t - 9$, $3t^2 - 8t + 2$, and $2t^2 - 3t + 1$
10. $11s^2 - 5s - 3$, $-4s^2 + 6s + 2$, and $-3s^2 - 8s + 5$
11. $2x^2 - 3xy + y^2$, $-3x^2 - 2y^2 + 5xy$, and $-2y^2 + 4x^2$
12. $5m^2 - mn - n^2$, $4n^2 - 2m^2 + 3mn$, and $-3n^2 - m^2$
13. $-pr + 4r^2$, $3p^2 - 2r^2$, and $6pr + 7p^2 - 5r^2$
14. $-10ab + 6a^2 - 5b^2$, $a^2 + 2ab - 3b^2$, and $-4ab + 5a^2$
15. $9c^2 - d^2 - 6cd$, $5cd - 4d^2$, and $3d^2 - 7c^2 + 2cd$

Test V — *Addition of Polynomials*

Time: 15 Minutes

Copy and add the polynomials in parts 1 to 15 of Table II.

Test VI — *Subtraction of Polynomials*

Time: 10 Minutes

For this test, subtract the first polynomial from the second in each of the fifteen parts of Table II. Copy the polynomials before subtracting.

Table III (Used in Tests VII, VIII)

1. $2w^2 - 11w + 5;\ 2w - 1$

2. $3b^2 - 19b + 6;\ b - 6$

3. $6m^2 - 11m + 3;\ 2m - 3$

4. $21A^2 + 2A - 8;\ 3A + 2$

5. $10x^2 - 13x - 30;\ 2x - 5$

6. $5c^2 - 23cd + 12d^2;\ 5c - 3d$

7. $10c^2 + 9cd - 9d^2;\ 2c + 3d$

8. $4x^2 - 15y^2 + 4xy;\ 2x + 5y$

9. $4a^2 - 21b^2 - 8ab;\ 2a - 7b$

10. $9r^2 - 35s^2 - 6rs;\ 5s + 3r$

11. $x^3 - 5x^2 + 9x - 9;\ x - 3$

12. $4a^3 + 8a^2 - 39a + 27;\ 2a - 3$

13. $9c^3 - 34c - 9c^2 - 16;\ 3c + 2$

14. $2m^3 - 7m^2 - m + 15;\ m^2 - m - 3$

15. $2x^3 - 2x^2y + xy^2 - 6y^3;\ 2x^2 + 2xy - 3y^2$

Test VII — *Multiplication of Polynomials*

Time: 15 Minutes

Copy and multiply the first polynomial by the second in parts 1 to 15 of Table III. Fifteen examples.

Test VIII — *Division of Polynomials* — Time: 15 Minutes

Copy and divide the first polynomial by the second in each of the fifteen parts of Table III. Fifteen examples.

Test IX — *Special Products* — Time: 5 Minutes

Do not copy the factors. Write only the products.

1. $(3x + 2y)(3x - 2y)$
2. $(2a - 3b)(2a - 3b)$
3. $(9x - 2y)(x + y)$
4. $(x^2 - 5y)(x^2 + 5y)$
5. $(6s - 5t)(2s + 3t)$
6. $(3t + 4n)(3t + 4n)$
7. $(9ab - 4c)(9ab + 4c)$
8. $(7r^2 + 8s)(3r^2 - 2s)$
9. $(x - 10y)(x + 5y)$
10. $(7 - 6xy)(5 + 6xy)$
11. $(2 + 9x)(3 - 2x)$
12. $(10x^3 - 3y^2)(2x^3 + 4y^2)$
13. $(9mn^2 + 4)(6mn^2 - 3)$
14. $(x^2 + 7yz)(5x^2 - 4yz)$
15. $(7a^2 + 3b^2)(5a^2 - 2b^2)$
16. $(3c^2 - 5d)(4c^2 - 5d)$
17. $(20a - 7b)(8a + 3b)$
18. $(10m - n)(6m + 5n)$
19. $(8 - 3x^2)(10 + x^2)$
20. $(14rs - 9)(2rs + 1)$

Test X — *Factoring* — Time: 5 Minutes

Do not copy the example. Factor.

1. $2m^3 - 10m^2$
2. $16x^2 - 25y^2$
3. $z^2 - 12z + 35$
4. $8a^2 - 22a + 15$
5. $2x^2 + x - 15$
6. $9r^2 - 6r - 8$
7. $15c^2 - 4c - 3$
8. $r^2s - 4rs - 45s$
9. $4x^2 - 24x + 35$
10. $3z^2 + 22z + 7$
11. $m - 36ma^2b^2$
12. $4a^2 - 28ax + 49x^2$
13. $x^2 + 7xy - 60y^2$
14. $6x^2 + 31x + 35$
15. $12a^2 + 13a - 35$
16. $2x^2 - 3xy - 20y^2$
17. $s^2 - 6st - 55t^2$
18. $10a^2 + 9ab - 9b^2$
19. $81x^2 - 72xy + 16y^2$
20. $4x^2 - 8xy - 21y^2$

TEST XI — *Simplification*

Time: 10 Minutes

Copy the examples. Remove parentheses and combine terms.

1. $3(2x-4)+2(x-5)$
2. $6(2-3x)-3(4x-7)$
3. $-2(a-6)+5(3a-2)$
4. $-2(t+9)-3(t-4)$
5. $-4(c-3)-3(2c-5)$
6. $2x(x-3y)+y(x+2y)$
7. $3a(2a-4b)-b(a-5b)$
8. $-5r(s-2r)+2s(r-3s)$
9. $-4c(3c+2d)-c(2c-d)$
10. $8m(m-3n)-5n(2m+n)$
11. $(x-4)(x+6)-(x+5)(x-2)$
12. $(y+3)(2y-1)+(3y-2)(y-4)$
13. $(2x-1)(x+3)-(3x+2)(x-1)$
14. $2(x+2)(x-3)+3(x-4)(x+5)$
15. $3(2x+1)(x-2)-2(3x-2)(x-3)$

TEST XII — *Simple Equations*

Time: 10 Minutes

Copy the equation. Solve. Omit check.

1. $5a+5=61-3a$
2. $\frac{x}{2}-\frac{x}{3}=\frac{5}{6}$
3. $4(y-5)+7=15$
4. $9m-7=3m-37$
5. $\frac{2s}{3}=\frac{s}{4}+5$
6. $3-(x-3)=5-2x$
7. $13-6x=13x-6$
8. $m-\frac{3m}{5}=9-\frac{m}{2}$
9. $3(x+1)=10(x-6)$
10. $7t+10=16t-17$
11. $\frac{c}{2}-\frac{5}{3}=\frac{3c}{4}-\frac{2c}{3}$
12. $4c=6(3-4c)-7(4+c)$
13. $15-6n=5n+48$
14. $\frac{4t}{9}-\frac{5t}{6}=\frac{2}{3}-\frac{3t}{2}$
15. $(3x+4)-(2x+9)=15$
16. $13+4p=11p-22$
17. $\frac{2z}{3}-\frac{9}{8}=\frac{5z}{6}-\frac{5}{4}$
18. $3n-2(2n-7)=3(n-2)$

TEST XIII — *Fractional Equations*

Time: 15 Minutes

Copy the equations. Solve them.

1. $\frac{x+5}{2} - \frac{x+1}{4} = 3$
2. $\frac{6}{a} - \frac{1}{a} = \frac{5}{3}$
3. $\frac{12}{x-2} = 3$
4. $\frac{11+m}{6} - \frac{10-m}{3} = 1$
5. $\frac{x-2}{5x} = \frac{4}{x} - 2$
6. $\frac{10}{m-3} = \frac{9}{m-5}$
7. $\frac{a+5}{3} - 4 = \frac{a-10}{4}$
8. $\frac{5t+4}{2t} - \frac{11t-2}{6t} = 3$
9. $\frac{x+5}{x-3} = 5 - \frac{4}{x-3}$
10. $\frac{c-2}{4} - \frac{c-4}{6} = \frac{2}{3}$
11. $\frac{r-3}{2r} = \frac{1}{3} - \frac{3r-7}{2r}$
12. $\frac{5x+1}{2x-3} = \frac{x}{2x-3} - 3$
13. $\frac{t-1}{3} + 3 = \frac{t+14}{9}$
14. $\frac{12}{x} - \frac{2x+8}{5x} + 3 = 0$
15. $\frac{4-x}{1-x} = \frac{12}{3-x} + 1$
16. $\frac{7w+1}{4} - \frac{w-9}{4} = 1$
17. $\frac{3m-5}{4} - \frac{9m-7}{12} = \frac{2}{3m}$
18. $\frac{2}{x+2} - \frac{5}{x-2} = \frac{2}{x^2-4}$

TEST XIV — *Simultaneous Equations*

Time: 15 Minutes

Solve the odd ones by the addition-subtraction method of elimination; the even ones by the substitution method.

1. $\begin{cases} 3x + y = 11 \\ 5x - y = 13 \end{cases}$
2. $\begin{cases} 3m + 2n = 17 \\ 4m + n = 16 \end{cases}$
3. $\begin{cases} 4a + 3b = -1 \\ 5a + b = 7 \end{cases}$
4. $\begin{cases} r - 6s = 2 \\ 3s - 8r = 29 \end{cases}$

5. $\begin{cases} r - 6s = -10 \\ 2r - 7s = -15 \end{cases}$

6. $\begin{cases} 5x + 8y = 2 \\ 10x - 12y = 32 \end{cases}$

7. $\begin{cases} 5m + 4n = 22 \\ 3m + n = 9 \end{cases}$

8. $\begin{cases} 2x + 5y = 13 \\ 7x - 4y = -19 \end{cases}$

9. $\begin{cases} 7c - 2d = 31 \\ 4c - 3d = 27 \end{cases}$

10. $\begin{cases} 8x + 5y = 18 \\ 16x - 3y = 10 \end{cases}$

11. $\begin{cases} 5a + 3b = -9 \\ 3a - 4b = -17 \end{cases}$

12. $\begin{cases} 8p + 5q = 5 \\ 3p - 2q = 29 \end{cases}$

13. $\begin{cases} 6x + 2y = -3 \\ 5x - 3y = -6 \end{cases}$

14. $\begin{cases} 5m - 12n = -31 \\ 3m + 22n = -4 \end{cases}$

15. $\begin{cases} 3s + 7t = 4 \\ 7s + 8t = 26 \end{cases}$

16. $\begin{cases} 2x - 3y = -14 \\ 3x + 7y = 48 \end{cases}$

TEST XV — *Quadratic Equations*

Time: 15 Minutes

Solve the odd ones by the factoring method; the even ones by completing the square or by the formula as your teacher directs.

1. $x^2 - 36 = 0$

2. $x^2 - 5x + 6 = 0$

3. $x^2 + 2x - 24 = 0$

4. $2x^2 - 11x + 5 = 0$

5. $2w^2 + 9w + 4 = 0$

6. $2y^2 - 3y - 5 = 0$

7. $3m^2 + 4m - 7 = 0$

8. $r^2 = 144$

9. $x^2 = 9x - 20$

10. $y^2 - 4y = 45$

11. $8a^2 = 22a - 15$

12. $12t^2 + 13t + 3 = 0$

13. $9r^2 - 6r = 8$

14. $2x^2 = 15 - x$

15. $4r^2 - 25 = 0$

16. $x^2 + 35 = 12x$

17. $y^2 = 7y + 18$

18. $5c^2 + 12 = 23c$

19. $2m^2 - 3m = 20$

20. $15c^2 = 4c + 3$

21. $x^2 = 21 - 4x$

22. $3x^2 = 10 - x$

23. $6y^2 - 7y = 3$

24. $8t^2 + 2t = 15$

DEFINITIONS, RULES, AND PROCESSES YOU SHOULD KNOW NOW

I. **Definitions.**— Be able to give the meaning of each of the following words or phrases, either as given in the book (which is preferable, when a definition is given in the book), or in your own words.

1. Literal number
2. Product
3. Formula
4. Equation
5. Factors of a number
6. Ray
7. Angle
8. Acute angle
9. Obtuse angle
10. Complementary angles
11. Complement
12. Supplementary angles
13. Supplement
14. Straight angle
15. Perimeter
16. Algebraic expression
17. Power of a number
18. Base (of a power)
19. Exponent
20. Absolute value
21. Monomial or term
22. Coefficient
23. Like terms
24. Unlike terms
25. Polynomial
26. Binomial
27. Trinomial
28. Ascending powers
29. Descending powers
30. Sum
31. Minuend
32. Subtrahend
33. Remainder
34. Parentheses
35. Brackets
36. Identity
37. Multiplier
38. Multiplicand
39. Product
40. Conditional equation
41. First degree equation
42. To factor a number
43. Prime number
44. Square root of a number
45. Quadratic equation, or second degree equation
46. Root of an equation
47. Literal equation
48. Common factor
49. Numerator
50. Denominator
51. Multiple
52. Common multiple
53. Lowest common multiple
54. Integral expression
55. Mixed expression
56. Complex fraction
57. Ratio
58. Proportion
59. Variable
60. Horizontal axis
61. Vertical axis
62. Origin
63. Abscissa
64. Ordinate
65. Coördinates

66. Independent equations
67. Dependent equations
68. Simultaneous equations
69. Inconsistent equations
70. Surd
71. Pure quadratic equation
72. Complete quadratic
73. Integral polynomial
74. Rational polynomial
75. Conjugate surds
76. Imaginary number
77. Complex number
78. Imaginary unit
79. Rational number
80. Irrational number
81. Irrational equation
82. Logarithm
83. Mantissa
84. Characteristic
85. Arithmetic progression
86. Arithmetic means
87. Geometric progression
88. Geometric means
89. Binomial theorem

II. Rules and facts you should be able to give.

1. Formula for the area of
    a. a rectangle;
    b. a parallelogram;
    c. a triangle;
    d. a trapezoid;
    e. a circle.
2. Formula for the length or circumference of a circle.
3. Formula for the volume of a rectangular parallelopiped.
4. Four fundamental rules used in solving a simple equation.
5. What is the sum of the angles of a triangle?
6. What is the sum of all the angles around a point?
7. The order in which fundamental operations are to be performed. (§ 22)
8. Rule for adding numbers having
    a. like signs; *b.* unlike signs.
9. Rule for multiplying numbers having:
    a. like signs; *b.* unlike signs.
10. Rule for adding like terms.
11. Rule for subtraction in algebra.
12. Rules for removing parentheses.
13. Law of exponents for multiplication.
14. Law of exponents for division.
15. Two fundamental principles of fractions.
16. Rule for adding or subtracting fractions.
17. Rule for multiplying fractions.
18. Rule for dividing by a fraction.

III. Examples and problems you should be able to solve and check.

1. How to add polynomials.
2. How to subtract one polynomial from another.
3. How to multiply one polynomial by another.
4. How to divide one polynomial by another.
5. How to multiply mentally binomials of the form $ax + b$, as taught on page 128.
6. How to factor a polynomial of the form $ax^2 + bx + c$, as taught on page 130.
7. How to add and subtract fractions.
8. How to multiply and divide fractions.
9. How to solve a simple equation like the ones on page 107.
10. How to solve a fractional equation like:
    a. numbers 1–16, on page 187;
    b. numbers 18–27, on page 188;
    c. numbers 29–54, on page 188.
11. How to draw the graph of a simple equation, as on page 216.
12. How to solve a system of two linear equations graphically, as on page 221.
13. How to solve an equation by elimination:
    a. by addition or subtraction, as on page 226;
    b. by substitution, as on page 229.
14. How to solve simultaneous fractional equations, as on page 231.
15. How to get the square root of a number:
    a. by factoring, as on page 248;
    b. by long division, as on pages 252–254.
16. How to solve a pure quadratic, as on page 261.
17. How to solve a complete quadratic:
    a. by factoring, as on pages 148 and 150;
    b. by completing the square, as on page 266;
    c. by formula, as on page 271.
18. How to form the equation for:
    a. a "number relations" problem;
    b. an "age" problem;
    c. a "distance, rate, and time" problem;
    d. a "money" problem;
    e. a "river" problem;
    f. an "interest" problem;
    g. a "digits" problem;

  h. an "area" problem;
  i. a "perimeter" problem;
  j. problems about angles;
  k. problems about fractions.

19. How to factor $x^2 - y^2 + 2yz - z^2$. (291)
20. How to factor $x^4 + x^2y^2 + y^4$. (292)
21. How to factor by grouping. (293)
22. How to factor $x^n \pm y^n$. (294)
23. How to factor by the factor theorem. (301)
24. How to draw the graph of a quadratic equation having two unknowns. (309)
25. How to solve graphically a system involving one or two quadratics. (313)
26. How to solve by elimination a system consisting:
    a. of a linear and a quadratic; (316)
    b. of two quadratics. (317)
27. How to tell the kind of roots a given quadratic has, by examination of the discriminant.
28. How to find the sum and the product of the roots of a given quadratic.
29. How to reduce a surd to its simplest form. (346)
30. How to change a surd to an equivalent surd of different order. (348)
31. How to multiply by (349) or to divide by (351–353) a surd.
32. How to solve an irrational equation. (359)
33. How to find the logarithm of a number (369) and to find the number when its logarithm is given. (372–374)
34. How to find the *n*th term and the sum of the first *n* terms of:
    a. an arithmetic progression; (381, 383)
    b. a geometric progression. (388, 389)
35. How to expand a power of a binomial by the binomial theorem. (397)

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